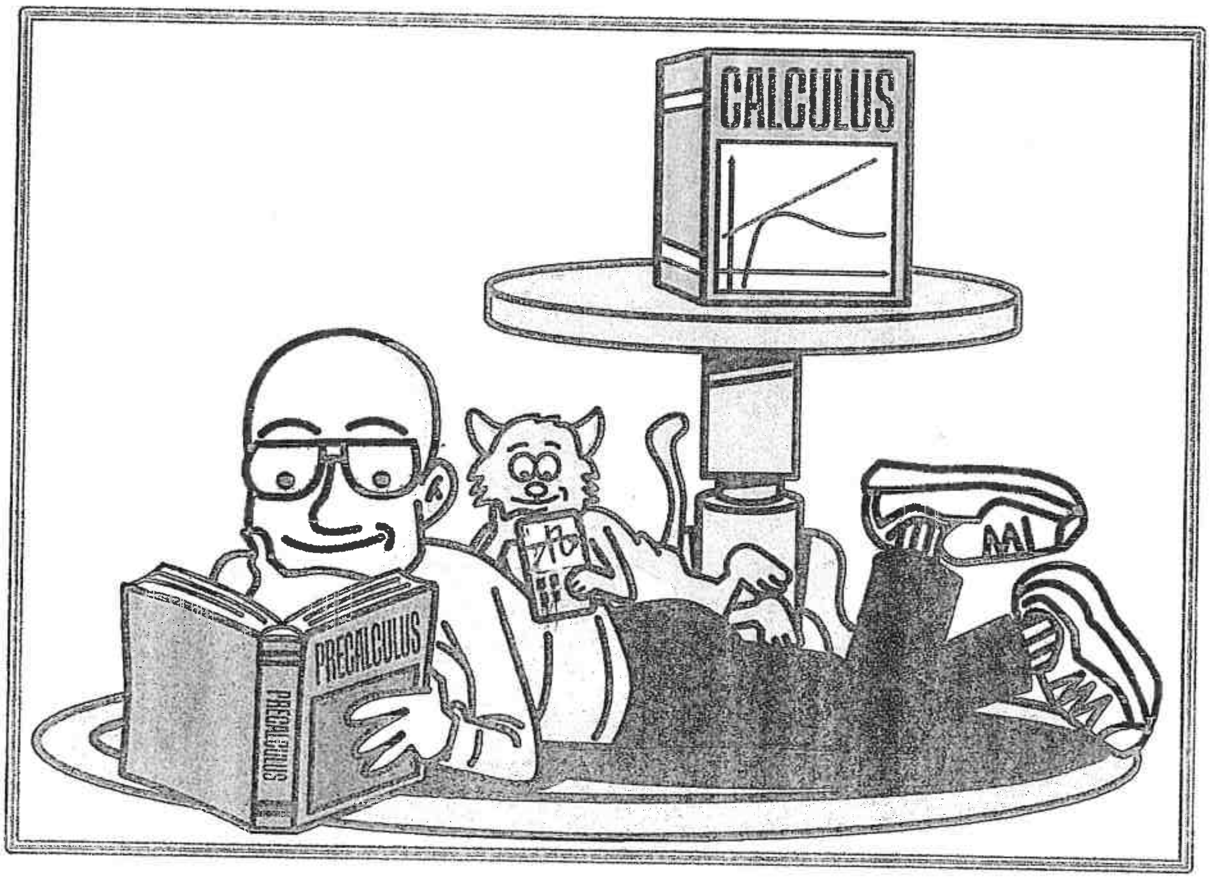
RU Ready for Some Calculus?

A Precalculus Review



Student Version

RU for Some Calculus? A Precalculus Review

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To the Student:

Picture a block of Swiss cheese. It is filled with holes and yet it stays in one piece. But this block has to be cut into slices. If there are too many holes, the slice will simply fall apart.

So it is with calculus. The AP Calculus course you are about to take is based on your foundation in mathematics – all the math that you have ever learned will come into play in this course. If you are taking calculus, it is possible that some of that material you knew fairly well at one time, but unfortunately, without everyday use, you just plain forget it. It is also possible that you never really learned it at all.

When you start your AP Calculus course, teachers make the assumption that you have mastered a lot of mathematics and techniques that you need to know are part of you. But it is a bad assumption and worse, a lot of teachers know it. In the past, teachers would start a new year by reviewing and getting everyone's knowledge at the same level. But in calculus, there is simply not enough time to spend time in review

So teachers teach AP Calculus knowing that there are extreme deficiencies in their student's math skills. And if the deficiencies are serious, the entire year crumbles like a piece of Swiss cheese with too many holes – holes in mathematical knowledge!

So what is the answer? Review all the math you have ever had? No, that just takes too long and who really cares enough to do that.

So this booklet contains all the material from precalculus that you really need to know going into AP calculus. It does not necessarily review the most difficult concepts of precalculus, but it takes the concepts that were in precalculus and are quite likely to show up in AP Calculus and teaches you, once and for all, to handle problems using those concepts.

For instance, the concept of complex fractions, fractions within fractions, usually only show up in precalculus when you are studying that concept. They rarely show up in word problems or in any other context. So you learn them when you need them, and you forget them 10 minutes after the test.

But in AP calculus, complex fractions occur fairly frequently Calculus is hard enough and if you lose points on a problem, you want it to be because you had a conceptual issue with the calculus topic, not because your knowledge of precalculus, specifically complex fractions, was faulty.

So this booklet contains just those concepts that are important for you in learning AP calculus. Topics like the conic sections, imaginary numbers, and finding rational zeros of functions, while important in precalculus, are rarely used in AP Calculus so they aren't included in this booklet.

You can be sure that if you review and master all the topics in this booklet, you are well on your way to doing well in AP Calculus. The reason is that many students worldwide struggle in AP calculus because their precalculus abilities are not good. Spending about eight hours on this booklet in total insures that is not going to happen to you!

RU Ready?

Let's talk about your calculus course. You are taking an Advanced Placement Calculus course. It is either AB Calculus or BC Calculus. Let's understand what these unusual names mean. The A.P Calculus program started in the year 1956. There was only one calculus exam given in these early years and it was called "Math." However, once the AP Calculus program got rolling fully, the courses were split into AB and BC and the first year there was a specific AB and BC exam was in the year 1969 There were three general topics into which all math problems fall

A Topics: these are precalculus concepts. They use no calculus but are considered necessary to understand and master before a student can master calculus.

B Topics. these are comprised of the calculus concepts taught in a first-year college calculus course.

C Topics: these are the calculus concepts taught in a second-year college calculus course.

So in a typical AB Calculus course, students will see problems including A topics and B topics and while in a BC course, students will see problems including B topics and C topics. Before the year 2000, there were problems on the AB exam that were strictly A topics no calculus was required. That is no longer true. In reality, all 45 multiple-choice questions and 6 free response questions on the AB exam are B topic questions. They are designed to test calculus.

So, although A topics are not specifically tested, students still need to understand them. You need to be able to solve equations, add algebraic fractions, find logarithms, and find trig functions of special angles. As with spelling, while students are not tested specifically on their spelling abilities by the time they get to high school, it is assumed that they know how to spell.

So, as a review, I have chosen 21 precalculus (A topics) that you really need to know and have mastered before you start your calculus book. This is not meant to be a complete review and if some of these topics are still a mystery to you, ask your teacher for an algebra, trigonometry, or precalculus book to borrow to sharpen your skills. The topics are not the only ones essential to mastering precalculus but were chosen because they crop up continuously in calculus examples. The way you see these examples expressed demonstrate how you will see them in calculus problems.

After every general topic and description, you will see sample problems with solutions worked out. On the back of each page, you will find roughly 12-15 problems that are similar to the examples. Your teachers might assign these over the summer It is suggested that you do one topic a day Your teacher might give you a 25-question multiple choice test the first day or so in class or give it to you as a summer take-home type exam. Please take it seriously Do well in this and you have mastered all the precalculus you need for AP Calculus. You will feel good that your block of Swiss cheese has few holes and when the block is sliced (and you break down calculus concepts), that the piece will stay together

Best of luck.

RU Ready?

A. Functions

The lifeblood of precalculus is functions. A **function** is a set of points (x, y) such that for every x, there is one and only one y In short, in a function, the x-values cannot repeat while the y-values can. In AB Calculus, all of your graphs will come from functions.

The notation for functions is either "$y=$" or "$f(x)=$" In the $f(x)$ notation, we are stating a rule to find y given a value of x.

1 If $f(x)=x^2-5x+8$, find a) $f(-6)$ b) $f\left(\frac{3}{2}\right)$ c) $\frac{f(x+h)-f(x)}{h}$

a) $f(-6)=(-6)^2-5(-6)+8$
$36+30+8$
74

b) $f\left(\frac{3}{2}\right)=\left(\frac{3}{2}\right)^2-5\left(\frac{3}{2}\right)+8$
$\frac{9}{4}-\frac{15}{2}+8$
$\frac{11}{4}$

c) $\frac{f(x+h)-f(x)}{h}$

$$\frac{(x+h)^2-5(x+h)+8-(x^2-5x+8)}{h}$$

$$\frac{x^2+2xh+h^2-5x-5h+8-x^2+5x-8}{5}$$

$$\frac{h^2+2xh-5h}{h}=\frac{h(h+2x-5)}{h}=h+2x-5$$

Functions do not always use the variable x. In calculus, other variables are used liberally

2. If $A(r)=\pi r^2$, find a) $A(3)$ b) $A(2s)$ c) $A(r+1)-A(r)$

$A(3)=9\pi$

$A(2s)=\pi(2s)^2=4\pi s^2$

$A(r+1)-A(r)=\pi(r+1)^2-\pi r^2$
$\pi(2r+1)$

One concept that comes up in AP calculus is **composition of functions** The format of a composition of functions is. plug a value into one function, determine an answer, and plug that answer into a second function.

3 If $f(x)=x^2-x+1$ and $g(x)=2x-1$, a) find $f(g(-1))$ b) find $g(f(-1))$ c) show that $f(g(x))\neq g(f(x))$

$g(-1)=2(-1)-1=-3$
$f(-3)=9+3+1=13$

$f(-1)=1+1+1=3$
$g(3)=2(3)-1=5$

$f(g(x))=f(2x-1)=(2x-1)^2-(2x-1)+1$
$=4x^2-4x+1-2x+1+1=4x^2-6x+3$
$g(f(x))=g(x^2-x+1)=2(x^2-x+1)-1$
$=2x^2-2x+1$

Finally, expect to use **piecewise functions**. A piecewise function gives different rules, based on the value of x

4 If $f(x)=\begin{cases} x^2-3, & x\geq 0 \\ 2x+1, & x<0 \end{cases}$, find a) $f(5)$ b) $f(2)-f(-1)$ c) $f(f(1))$

$f(5)=25-3=22$

$f(2)-f(-1)=1-(-1)=2$

$f(1)=-2,\ f(-2)=-3$

A. Function Assignment

- If $f(x)=4x-x^2$, find

1 $f(4)-f(-4)$

2. $\sqrt{f\left(\frac{3}{2}\right)}$

3 $\frac{f(x+h)-f(x-h)}{2h}$

- If $V(r)=\frac{4}{3}\pi r^3$, find

4 $V\left(\frac{3}{4}\right)$

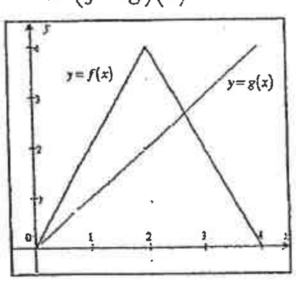
5 $V(r+1)-V(r-1)$

6 $\frac{V(2r)}{V(r)}$

- If $f(x)$ and $g(x)$ are given in the graph below, find

7. $(f-g)(3)$

8. $f(g(3))$



- If $f(x)=x^2-5x+3$ and $g(x)=1-2x$, find

9 $f(g(x))$

- If $f(x)=\begin{cases}\sqrt{x+2}-2, & x\geq 2\\ x^2-1, & 0\leq x<2\\ -x, & x<0\end{cases}$, find

10. $f(0)-f(2)$

11 $\sqrt{5-f(-4)}$

12. $f(f(3))$

RU Ready?

B. Domain and Range

First, since questions in calculus usually ask about behavior of functions in intervals, understand that intervals can be written with a description in terms of $<, \le, >, \ge$ or by using **interval notation**.

Description	Interval notation	Description	Interval notation	Description	Interval notation
$x > a$	(a,∞)	$x \le a$	$(-\infty,a]$	$a \le x < b$	$[a,b)$
$x \ge a$	$[a,\infty)$	$a < x < b$	(a,b) · open interval	$a < x \le b$	$(a,b]$
$x < a$	$(-\infty,a)$	$a \le x \le b$	$[a,b]$ · closed interval	All real numbers	$(-\infty,\infty)$

If a solution is in one interval or the other, interval notation will use the connector $\cup$ So $x \le 2$ or $x > 6$ would be written $(-\infty,2]\cup(6,\infty)$ in interval notation. Solutions in intervals are usually written in the easiest way to define it. For instance, saying that $x < 0$ or $x > 0$ or $(-\infty,0)\cup(0,\infty)$ is best expressed as $x \ne 0$

The **domain of a function** is the set of allowable x-values. The domain of a function f is $(-\infty,\infty)$ except for values of x which create a zero in the denominator, an even root of a negative number or a logarithm of a non-positive number The domain of $a^{p(x)}$ where a is a positive constant and $p(x)$ is a polynomial is $(-\infty,\infty)$

• Find the domain of the following functions using interval notation.

1 $f(x) = x^2 - 4x + 4$ $\boxed{(-\infty,\infty)}$

2. $y = \dfrac{6}{x-6}$ $\boxed{x \ne 6}$

3 $y = \dfrac{2x}{x^2 - 2x - 3}$ $\boxed{x \ne -1, x \ne 3}$

4 $y = \sqrt{x+5}$ $\boxed{[-5,\infty)}$

5 $y = \sqrt[3]{x+5}$ $\boxed{(-\infty,\infty)}$

6. $y = \dfrac{x^2 + 4x + 6}{\sqrt{2x+4}}$ $\boxed{(-2,\infty)}$

The **range of a function** is the set of allowable y-values. Finding the range of functions algebraically isn't as easy (it really is a calculus problem), but visually, it is the [lowest possible y-value, highest possible y-value]. Finding the range of some functions are fairly simple to find if you realize that the range of $y = x^2$ is $[0,\infty)$ as any positive number squared is positive. Also the range of $y = \sqrt{x}$ is also positive as the domain is $[0,\infty)$ and the square root of any positive number is positive. The range of $y = a^x$ where a is a positive constant is $(0,\infty)$ as constants to powers must be positive.

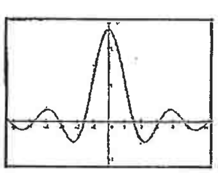
• Find the range of the following functions using interval notation.

7 $y = 1 - x^2$ $\boxed{(-\infty,1]}$

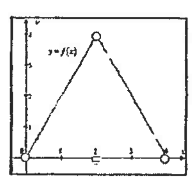
8 $y = \dfrac{1}{x^2}$ $\boxed{(0,\infty)}$

9 $y = \sqrt{x-8} + 2$ $\boxed{[2,\infty)}$

• Find the domain and range of the following functions using interval notation.

10 

Domain: $(-\infty,\infty)$
Range: $[-0.5,2.5]$

11 

Domain: $(0,4)$
Range: $[0,4)$

B. Domain and Range Assignment

- Find the domain of the following functions using interval notation.

1 $f(x)=3$

2. $y=x^3-x^2+x$

3 $y=\frac{x^3-x^2+x}{x}$

4 $y=\frac{x-4}{x^2-16}$

5 $f(x)=\frac{1}{4x^2-4x-3}$

6 $y=\sqrt{2x-9}$

7 $f(t)=\sqrt{t^3+1}$

8. $f(x)=\sqrt[5]{x^2-x-2}$

9 $y=5^{x^2-4x-2}$

10 $y=\log(x-10)$

11 $y=\frac{\sqrt{2x+14}}{x^2-49}$

12. $y=\frac{\sqrt{5-x}}{\log x}$

Find the range of the following functions.

13 $y=x^4+x^2-1$

14. $y=100^x$

15 $y=\sqrt{x^2+1}+1$

Find the domain and range of the following functions using interval notation.

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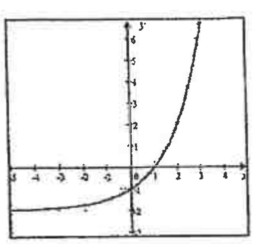

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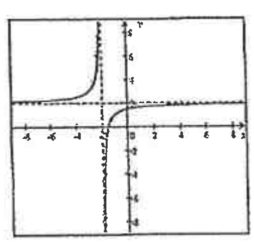

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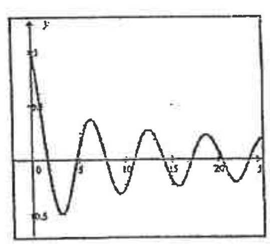
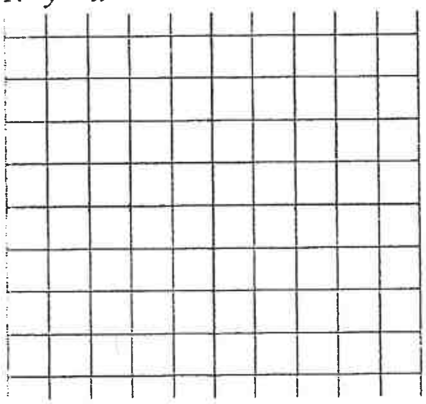


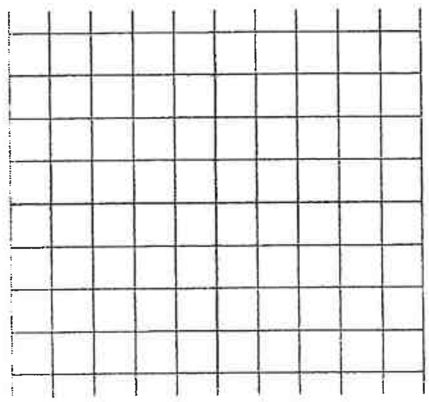
C. Graphs of Common Functions - Assignment

Sketch each of the following as accurately as possible. You will need to be VERY familiar with each of these graphs throughout the year You may use a graphing calculator if you have access to one over the summer [redacted]
Another option is to find a graphing app (there are free ones) or generate a table of values on your scientific calculator Again these are VERY important graphs to know Be very accurate with regards to "open circles" and "closed circles."

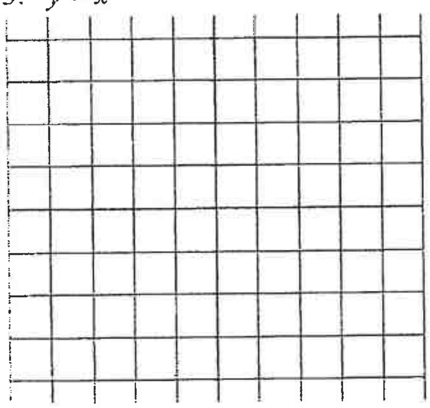
1. $y = x$



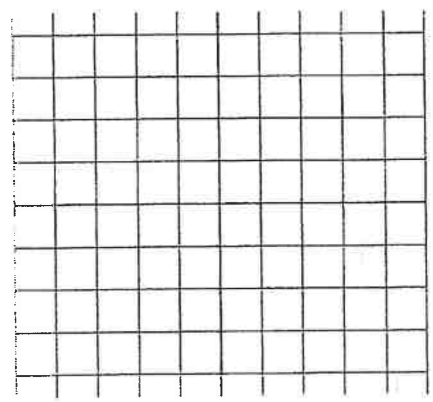
2. $y = x^2$



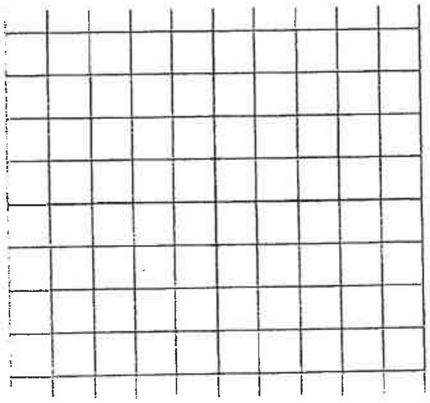
3. $y = x^3$



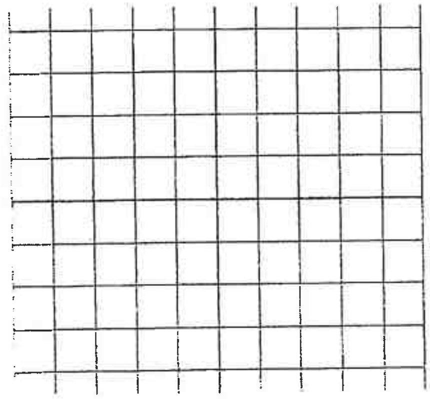
4 $y = \sqrt{x}$



5. $y = |x|$

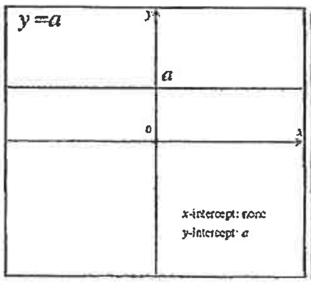


6 $y = \frac{|x|}{x}$



C. Graphs of Common Functions

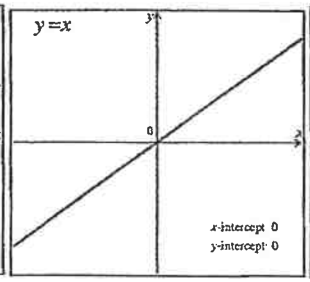
There are certain graphs that occur all the time in calculus and students should know the general shape of them, where they hit the x-axis (zeros) and y-axis (y-intercept), as well as the domain and range. There are no assignment problems for this section other than students memorizing the shape of all of these functions. In section 5, we will talk about transforming these graphs.



Function. $y = a$

Domain. $(-\infty,\infty)$

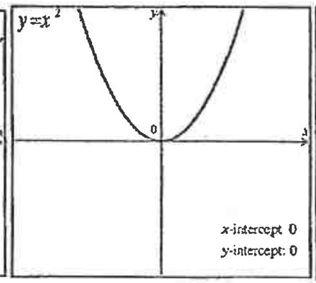
Range: $[a,a]$



Function. $y = x$

Domain. $(-\infty,\infty)$

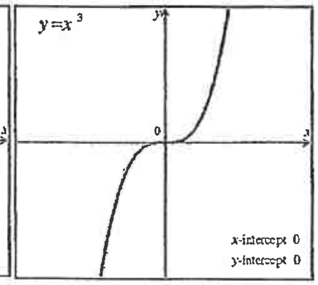
Range: $(-\infty,\infty)$



Function. $y = x^2$

Domain. $(-\infty,\infty)$

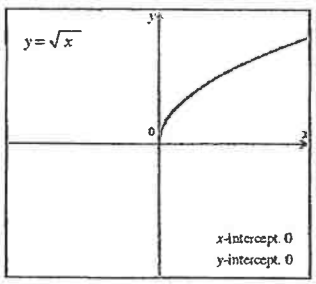
Range: $[0,\infty)$



Function. $y = x^3$

Domain $(-\infty,\infty)$

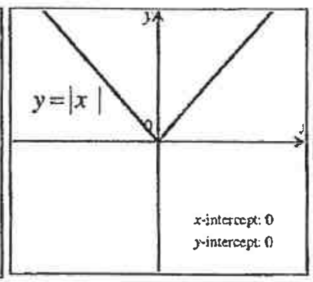
Range: $(-\infty,\infty)$



Function: $y = \sqrt{x}$

Domain. $[0,\infty)$

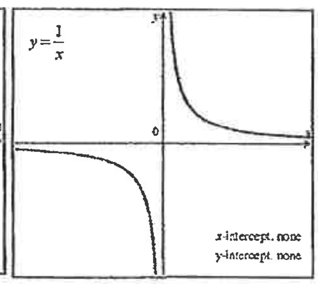
Range: $[0,\infty)$



Function. $y = |x|$

Domain. $(-\infty,\infty)$

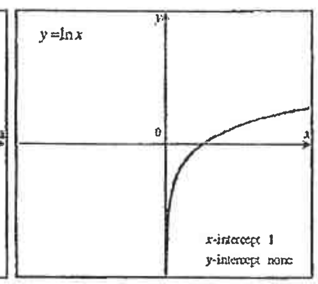
Range: $[0,\infty)$



Function $y = \dfrac{1}{x}$

Domain. $x \neq 0$

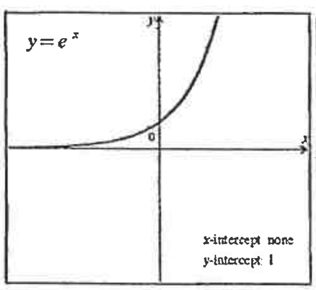
Range: $y \neq 0$



Function: $y = \ln x$

Domain. $(0,\infty)$

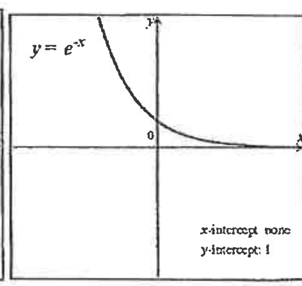
Range: $(-\infty,\infty)$



Function: $y = e^x$

Domain. $(-\infty,\infty)$

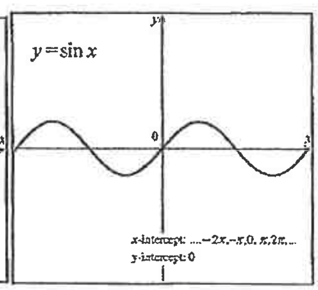
Range: $(0,\infty)$



Function. $y = e^{-x}$

Domain. $(-\infty,\infty)$

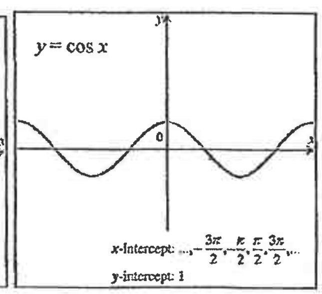
Range. $(0,\infty)$



Function: $y = \sin x$

Domain. $(-\infty,\infty)$

Range: $[-1,1]$

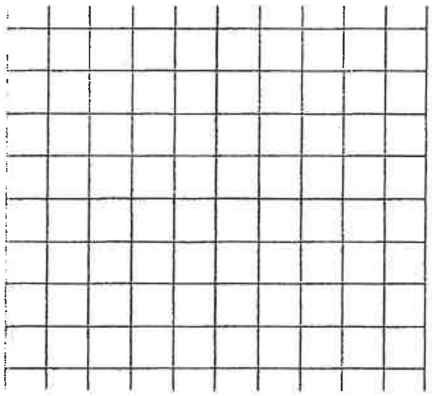


Function. $y = \cos x$

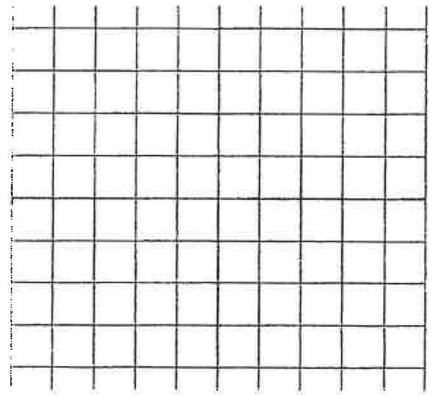
Domain: $(-\infty,\infty)$

Range: $[-1,1]$

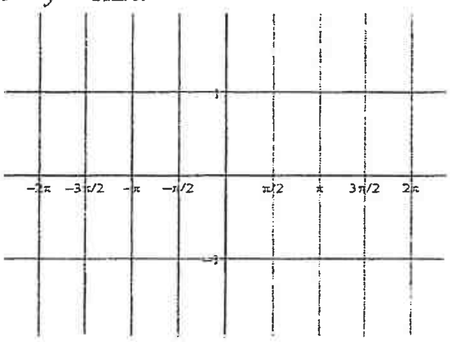
7 $y = x^{1/3}$



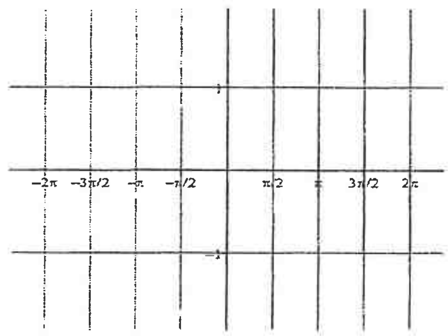
8 $y = x^{2/3}$



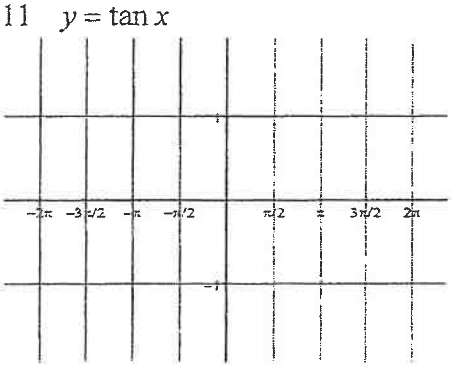
9 $y = \sin x$



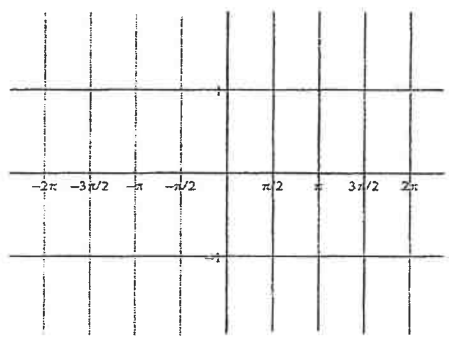
10. $y = \cos x$



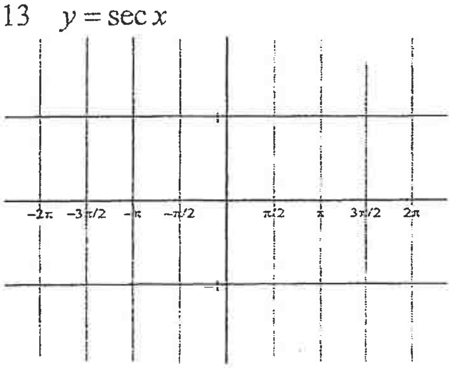
11 $y = \tan x$



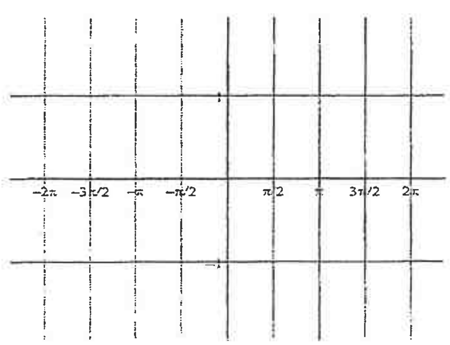
12. $y = \cot x$



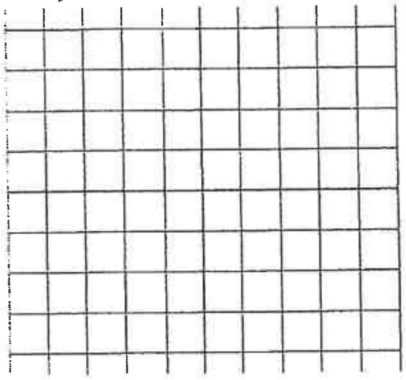
13 $y = \sec x$



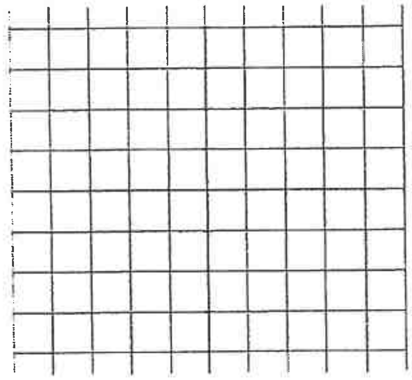
14. $y = \csc x$



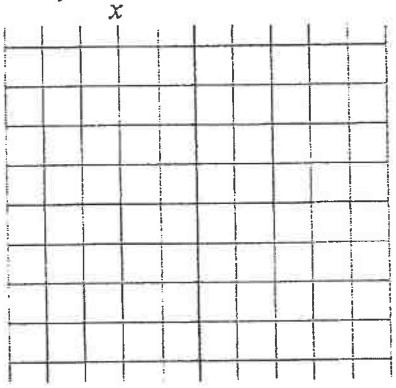
15 $y = e^x$



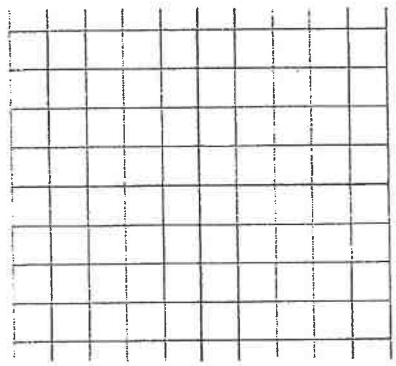
16 $y = \ln x$



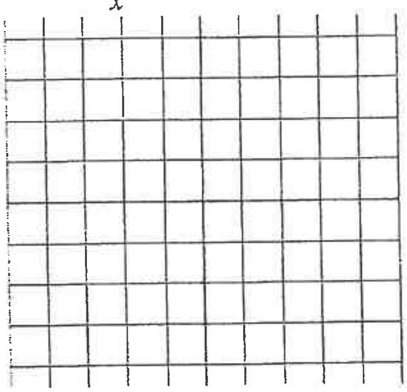
17 $y = \frac{1}{x}$



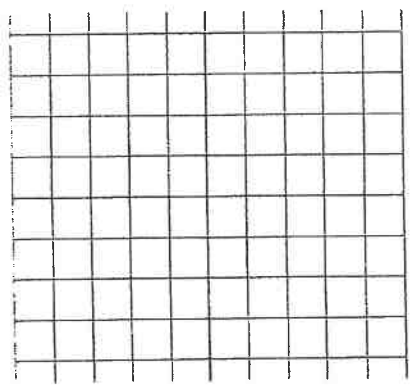
18 $y = [\![x]\!]$



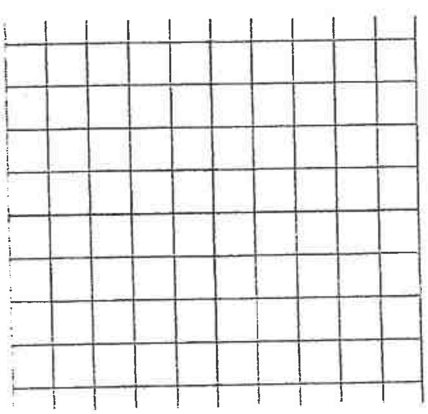
19 $y = \frac{1}{x^2}$



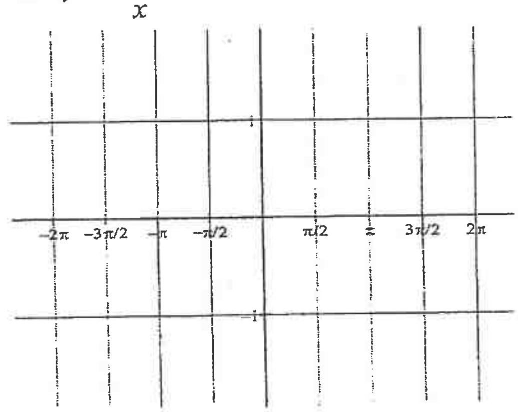
20. $y = 2^x$



21 $y = \sqrt{4 - x^2}$



22. $y = \frac{\sin x}{x}$



D. Even and Odd Functions

Functions that are even have the characteristic that for all a, $f(-a)=f(a)$ What this says is that plugging in a positive number a into the function or a negative number $-a$ into the function makes no difference you will get the same result. Even functions are symmetric to the y-axis.

Functions that are odd have the characteristic that for all a, $f(-a)=-f(a)$ What this says is that plugging in a negative number $-a$ into the function will give you the same result as plugging in the positive number and taking the negative of that. So, odd functions are symmetric to the origin. If a graph is symmetric to the x-axis, it is not a function because it fails the vertical-line test.

1 Of the common functions in section 3, which are even, which are odd, and which are neither?

Even. $y=a$, $y=x^2$, $y=|x|$, $y=\cos x$ Odd: $y=x$, $y=x^3$, $y=\frac{1}{x}$, $y=\sin x$

Neither: $y=\sqrt{x}$, $y=\ln x$, $y=e^x$, $y=e^{-x}$

2. Show that the following functions are even.

a) $f(x)=x^4-x^2+1$

$f(-x)=(-x)^4-(-x)^2+1$
$=x^4-x^2+1=f(x)$

b) $f(x)=\left|\frac{1}{x}\right|$

$f(-x)=\left|\frac{1}{-x}\right|=\left|\frac{1}{x}\right|=f(x)$

c) $f(x)=x^{2/3}$

$f(-x)=(-x)^{2/3}=\left(\sqrt[3]{-x}\right)^2$
$=\left(\sqrt[3]{x}\right)^2=f(x)$

3 Show that the following functions are odd.

a) $f(x)=x^3-x$

$f(-x)=(-x)^3+x$
$=x-x^3=-f(x)$

b) $f(x)=\sqrt[3]{x}$

$f(-x)=\sqrt[3]{-x}=-\sqrt[3]{x}=-f(x)$

c) $f(x)=e^x-e^{-x}$

$f(-x)=e^{-x}-e^x=-(e^x-e^{-x})=-f(x)$

4. Determine if $f(x)=x^3-x^2+x-1$ is even, odd, or neither Justify your answer

$f(-x)=-x^3-x^2-x-1\neq f(x)$ so f is not even. $-f(x)=-x^3+x^2-x-1\neq f(-x)$ so f is not odd.

Graphs may not be functions and yet have x-axis or y-axis or both. Equations for these graphs are usually expressed in "implicit form" where it is not expressed as "$y=$" or "$f(x)=$" If the equation does not change after making the following replacements, the graph has these symmetries:

x-axis. y with $-y$ y-axis: x with $-x$ origin. both x with $-x$ and y with $-y$

5 Determine the symmetry for $x^2+xy+y^2=0$

$x-$axis. $x^2+x(-y)+(-y)^2=0\Rightarrow x^2-xy+y^2=0$ so not symmetric to $x-$axis

$y-$axis: $(-x)^2+(-x)(y)+y^2=0\Rightarrow x^2-xy+y^2=0$ so not symmetric to $y-$axis

origin: $(-x)^2+(-x)(-y)+y^2=0\Rightarrow x^2+xy+y^2=0$ so symmetric to origin

D. Even and odd functions - Assignment

- Show work to determine if the following functions are even, odd, or neither

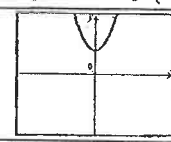
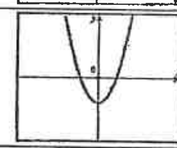
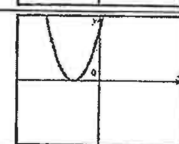
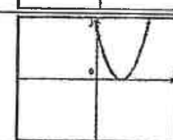
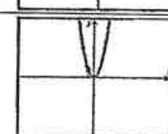
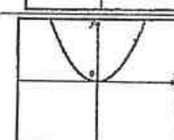
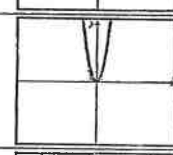
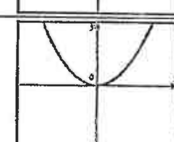
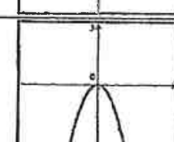
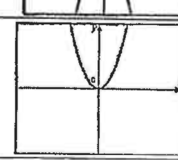
1. $f(x)=7$
2. $f(x)=2x^2-4x$
3. $f(x)=-3x^3-2x$
4. $f(x)=\sqrt{x+1}$
5. $f(x)=\sqrt{x^2+1}$
6. $f(x)=8x$
7. $f(x)=8x-\frac{1}{8x}$
8. $f(x)=|8x|$
9. $f(x)=|8x|-8x$

Show work to determine if the graphs of these equations are symmetric to the x-axis, y-axis or the origin.

10. $4x=1$
11. $y^2=2x^4+6$
12. $3x^2=4y^3$
13. $x=|y|$
14. $|x|=|y|$
15. $|x|=y^2+2y+1$

E. Transformation of Graphs

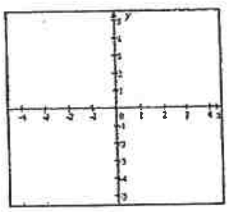
A curve in the form $y = f(x)$, which is one of the basic common functions from section C can be transformed in a variety of ways. The shape of the resulting curve stays the same but zeros and y-intercepts might change and the graph could be reversed. The table below describes transformations to a general function $y = f(x)$ with the parabolic function $f(x) = x^2$ as an example.

Notation	How $f(x)$ changes	Example with $f(x) = x^2$
$f(x)+a$	Moves graph up a units	
$f(x)-a$	Moves graph down a units	
$f(x+a)$	Moves graph a units left	
$f(x-a)$	Moves graph a units right	
$a\ f(x)$	$a > 1$ Vertical Stretch	
$a\ f(x)$	$0 < a < 1$ Vertical shrink	
$f(ax)$	$a > 1$ Horizontal compress (same effect as vertical stretch)	
$f(ax)$	$0 < a < 1$ Horizontal elongated (same effect as vertical shrink)	
$-f(x)$	Reflection across x-axis	
$f(-x)$	Reflection across y-axis	

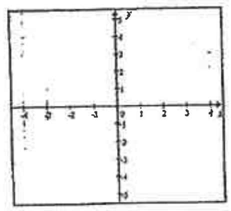
E. Transformation of Graphs Assignment

• Sketch the following equations:

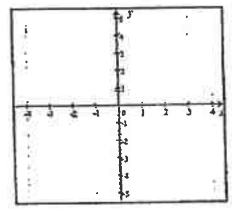
1 $y=-x^2$



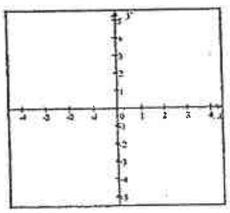
2. $y=2x^2$



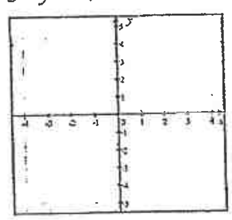
3. $y=(x-2)^2$



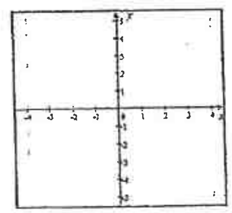
4 $y=2-\sqrt{x}$



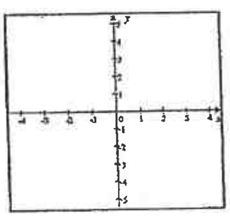
5 $y=\sqrt{x+1}+1$



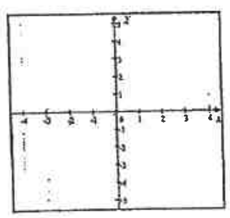
6. $y=\sqrt{4x}$



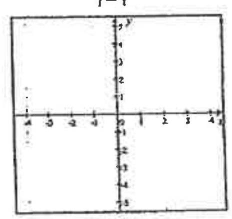
7 $y=|x+1|-3$



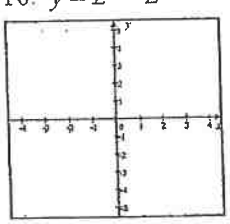
8. $y=-2|x-1|+4$



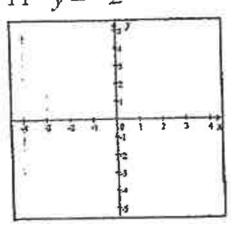
9. $y=-\left|\frac{x}{2}\right|-1$



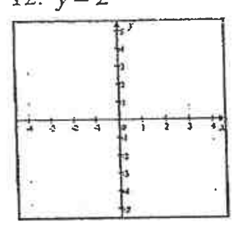
10. $y=2^x-2$



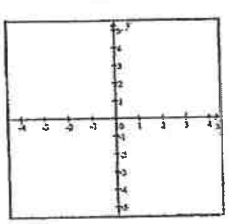
11 $y=-2^{x+2}$



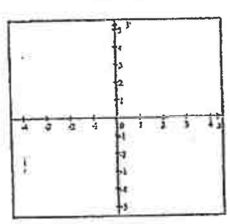
12. $y=2^{-2x}$



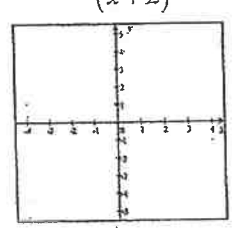
13 $y=\frac{1}{x-2}$



14. $y=\frac{-2}{x+1}$



15 $y=\frac{1}{(x+2)^2}-3$



F. Special Factorization

While factoring skills were more important in the days when A topics were specifically tested, students still must know how to factor The special forms that occur most regularly are:

Common factor: $x^3+x^2+x=x(x^2+x+1)$
Difference of squares: $x^2-y^2=(x+y)(x-y)$ or $x^{2n}-y^{2n}=(x^n+y^n)(x^n-y^n)$
Perfect squares: $x^2+2xy+y^2=(x+y)^2$
Perfect squares: $x^2-2xy+y^2=(x-y)^2$
Sum of cubes. $x^3+y^3=(x+y)(x^2-xy+y^2)$ - Trinomial unfactorable
Difference of cubes: $x^3-y^3=(x-y)(x^2+xy+y^2)$ - Trinomial unfactorable
Grouping: $xy+xb+ay+ab=x(y+b)+a(y+b)=(x+a)(y+b)$

The term "factoring" usually means that coefficients are rational numbers. For instance, x^2-2 could technically be factored as $(x+\sqrt{2})(x-\sqrt{2})$ but since $\sqrt{2}$ is not rational, we say that x^2-2 is not factorable. It is important to know that x^2+y^2 is unfactorable.

• Completely factor the following expressions.

1 $4a^2+2a$

$\boxed{2a(a+2)}$

2. $x^2+16x+64$

$\boxed{(x+8)^2}$

3 $4x^2-64$

$\boxed{4(x+4)(x-4)}$

4. $5x^4-5y^4$

$\boxed{5(x^2+1)(x+1)(x-1)}$

5 $16x^2-8x+1$

$\boxed{(4x-1)^2}$

6. $9a^4-a^2b^2$

$\boxed{a^2(3a+b)(3a-b)}$

7 $2x^2-40x+200$

$\boxed{2(x-10)^2}$

8. x^3-8

$\boxed{(x-2)(x^2+2x+4)}$

9 $8x^3+27y^3$

$\boxed{(2x+3y)(4x^2-6xy+9y^2)}$

10 x^4+11x^2-80

$\boxed{(x+4)(x-4)(x^2+5)}$

11 x^4-10x^2+9

$\boxed{(x+1)(x-1)(x+3)(x-3)}$

12. $36x^2-64$

$\boxed{4(3x+4)(3x-4)}$

13 x^3-x^2+3x-3

$x^2(x-1)+3(x-1)$

$(x-1)(x^2+3)$

14 $x^3+5x^2-4x-20$

$x^2(x+5)-4(x+5)$

$(x+5)(x-2)(x+2)$

15 $9-(x^2+2xy+y^2)$

$9-(x+y)^2$

$(3+x+y)(3-x-y)$

F. Special Factorization - Assignment

- Completely factor the following expressions

1 x^3-25x

2. $30x-9x^2-25$

3 $3x^2-5x^2+2x$

4 $3x^8-3$

5 $16x^4-24x^2y+9y^2$

6. $9a^4-a^2b^2$

7 $4x^4+7x^2-36$

8 $250x^3-128$

9 $\frac{8x^3}{125}+\frac{64}{y^3}$

10. x^5+17x^3+16x

11 $144+32x^2-x^4$

12. $16x^{4a}-y^{8a}$

13 $x^3-xy^2+x^2y-y^3$

14 $x^6-9x^4-81x^2+729$

15 $x^2-8xy+16y^2-25$

16 $x^5+x^3+x^2+1$

17 x^6-1

18. x^6+1

G. Linear Functions

Probably the most important concept from precalculus that is required for differential calculus is that of linear functions. The formulas you need to know backwards and forwards are:

Slope: Given two points (x_1, y_1) and (x_2, y_2), the slope of the line passing through the points can be written as·
$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$
Slope intercept form the equation of a line with slope m and y-intercept b is given by $y = mx + b$
Point-slope form: the equation of a line passing through the points (x_1, y_1) and slope m is given by $y - y_1 = m(x - x_1)$ While you might have preferred the simplicity of the $y = mx + b$ form in your algebra course, the $y - y_1 = m(x - x_1)$ form is far more useful in calculus.

Intercept form: the equation of a line with x-intercept a and y-intercept b is given by $\frac{x}{a} + \frac{y}{b} = 1$
General form $Ax + By + C = 0$ where A, B and C are integers. While your algebra teacher might have required your changing the equation $y - 1 = 2(x - 5)$ to general form $2x - y - 9 = 0$, you will find that on the AP calculus test, it is sufficient to leave equations for a lines in point-slope form and it is recommended not to waste time changing it unless you are specifically told to do so
Parallel lines Two distinct lines are parallel if they have the same slope: $m_1 = m_2$
Normal lines: Two lines are normal (perpendicular) if their slopes are negative reciprocals: $m_1 \; m_2 = -1$
Horizontal lines have slope zero. **Vertical lines** have no slope (slope is undefined).

1 Find the equation of the line in slope-intercept form, with the given slope, passing through the given point.

a. $m = -4, (1,2)$ b $m = \frac{2}{3}, (-5,1)$ c. $m = 0, \left(-\frac{1}{2}, \frac{3}{4}\right)$

$\boxed{y - 2 = -4(x-1) \Rightarrow y = -4x + 6}$ $\boxed{y - 1 = \frac{2}{3}(x-5) \Rightarrow y = \frac{2x}{3} - \frac{7}{3}}$ $\boxed{y = -\frac{3}{4}}$

2. Find the equation of the line in slope-intercept form, passing through the following points.

a. $(4,5)$ and $(-2,-1)$ b. $(0,-3)$ and $(-5,3)$ c. $\left(\frac{3}{4}, -1\right)$ and $\left(1, \frac{1}{2}\right)$

$\boxed{m = \frac{5+1}{4+2} = 1 \quad y - 5 = x - 4 \Rightarrow y = x + 1}$

$\boxed{m = \frac{3+3}{-5-0} = \frac{-6}{5} \quad y + 3 = \frac{-6}{5}x \Rightarrow y = \frac{-6}{5}x - 3}$

$\boxed{m = \left(\frac{\frac{1}{2}+1}{1-\frac{3}{4}}\right)\left(\frac{4}{4}\right) = \frac{2+4}{4-3} = 6 \quad y - \frac{1}{2} = 6(x-1) \Rightarrow y = 6x - \frac{11}{2}}$

3 Write equations of the line through the given point a) parallel and b) normal to the given line.

a. $(4,7)$, $4x - 2y = 1$ b. $\left(\frac{2}{3}, 1\right)$, $x + 5y = 2$

$\boxed{y = 2x - \frac{1}{2} \Rightarrow m = 2 \quad \text{a) } y - 7 = 2(x-4) \quad \text{b) } y - 7 = \frac{-1}{2}(x-4)}$

$\boxed{y = \frac{-1}{5}x + 2 \Rightarrow m = \frac{-1}{5} \quad \text{a) } y - 1 = \frac{-1}{5}\left(x - \frac{2}{3}\right) \quad \text{b) } y - 1 = 5\left(x - \frac{2}{3}\right)}$

G. Linear Functions - Assignment

1 Find the equation of the line in slope-intercept form, with the given slope, passing through the given point.

a. $m=-7, (-3,-7)$ b. $m=\frac{-1}{2}, (2,-8)$ c. $m=\frac{2}{3}, \left(-6,\frac{1}{3}\right)$

2 Find the equation of the line in slope-intercept form, passing through the following points.

a. $(-3,6)$ and $(-1,2)$ b $(-7,1)$ and $(3,-4)$ c. $\left(-2,\frac{2}{3}\right)$ and $\left(\frac{1}{2},1\right)$

3. Write equations of the line through the given point a) parallel and b) normal to the given line.

a. $(5,-3)$, $x+y=4$ b. $(-6,2)$, $5x+2y=7$ c. $(-3,-4)$, $y=-2$

4. Find an equation of the line containing $(4,-2)$ and parallel to the line containing $(-1,4)$ and $(2,3)$ Put your answer in general form.

5. Find k if the lines $3x-5y=9$ and $2x+ky=11$ are a) parallel and b) perpendicular.

H. Solving Quadratic Equations

Solving quadratics in the form of $ax^2+bx+c=0$ usually show up on the AP exam in the form of expressions that can easily be factored. But occasionally, you will be required to use the quadratic formula. When you have a quadratic equation, factor it, set each factor equal to zero and solve. If the quadratic equation doesn't factor or if factoring is too time-consuming, use the quadratic formula.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The **discriminant** b^2-4ac will tell you how many solutions the quadratic has:

$$b^2-4ac\begin{cases}>0, \text{ 2 real solutions (if a perfect square, the solutions are rational)}\\ =0, \text{ 1 real solution}\\ <0, \text{ 0 real solutions (or 2 imaginary solutions, but AP calculus does not deal with imaginaries)}\end{cases}$$

1 Solve for x

a. $x^2+3x+2=0$

$(x+2)(x+1)=0$
$x=-2, x=-1$

b $x^2-10x+25=0$

$(x-5)^2=0$
$x=5$

c. $x^2-64=0$

$(x-8)(x+8)=0$
$x=8, x=-8$

d. $2x^2+9x=18$

$(2x-3)(x+6)=0$
$x=\frac{3}{2}, x=-6$

e. $12x^2+23x=-10$

$(4x+5)(3x+2)=0$
$x=-\frac{5}{4}, x=-\frac{2}{3}$

f. $48x-64x^2=9$

$(8x-3)^2=0$
$x=\frac{3}{8}$

g. $x^2+5x=2$

$x=\frac{-5\pm\sqrt{25+8}}{2}$
$x=\frac{-5\pm\sqrt{33}}{2}$

h. $8x-3x^2=2$

$x=\frac{8\pm\sqrt{64-24}}{6}$
$x=\frac{8\pm2\sqrt{10}}{6}=\frac{4\pm\sqrt{10}}{3}$

i. $6x^2+5x+3=0$

$x=\frac{-5\pm\sqrt{25-72}}{12}=\frac{-5\pm\sqrt{-47}}{12}$
No real solutions

j $x^3-3x^2+3x-9=0$

$x^2(x-3)-3(x-3)=0$
$(x-3)(x^2-3)=0$
$x=3,\ x=\pm\sqrt{3}$

k. $\frac{x}{3}-\frac{5}{2}=\frac{-3}{x}$

$6x\left(\frac{x}{3}-\frac{5}{2}=\frac{-3}{x}\right)$
$2x^2-15x+18=0$
$(2x-3)(x-6)=0$
$x=\frac{3}{2},\ x=6$

l. $x^4-7x^2-8=0$

$(x^2-8)(x^2+1)=0$
$x=\pm\sqrt{8}=\pm2\sqrt{2}$

2 If $y=5x^2-3x+k$, for what values of k will the quadratic have two real solutions?

$(-3)^2-4(5)k>0\Rightarrow 9-20k>0\Rightarrow k<\frac{9}{20}$

H. Solving Quadratic Equations Assignment

1 Solve for x.

a. $x^2+7x-18=0$

b. $x^2+x+\frac{1}{4}=0$

c. $2x^2-72=0$

d. $12x^2-5x=2$

e. $20x^2-56x+15=0$

f. $81x^2+72x+16=0$

g. $x^2+10x=7$

h. $3x-4x^2=-5$

i. $7x^2-7x+2=0$

j. $x+\frac{1}{x}=\frac{17}{4}$

k. $x^3-5x^2+5x-25=0$

l. $2x^4-15x^3+18x^2=0$

2 If $y=x^2+kx-k$, for what values of k will the quadratic have two real solutions?

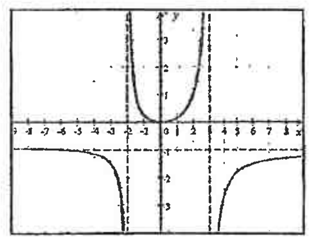
3 Find the domain of $y=\frac{2x-1}{6x^2-5x-6}$

I. Asymptotes

Rational functions in the form of $y=\frac{p(x)}{q(x)}$ possibly have vertical asymptotes, lines that the graph of the curve approach but never cross. To find the **vertical asymptotes**, factor out any common factors of numerator and denominator, reduce if possible, and then set the denominator equal to zero and solve.

Horizontal asymptotes are lines that the graph of the function approaches when x gets very large or very small. While you learn how to find these in calculus, a rule of thumb is that if the highest power of x is in the denominator, the horizontal asymptote is the line $y = 0$ If the highest power of x is both in numerator and denominator, the horizontal asymptote will be the line $y=\frac{\text{highest degree coefficient in numerator}}{\text{highest degree coefficient in denominator}}$ If the highest power of x is in the numerator, there is no horizontal asymptote, but a slant asymptote which is not used in calculus.

1) Find any vertical and horizontal asymptotes for the graph of $y=\frac{-x^2}{x^2-x-6}$

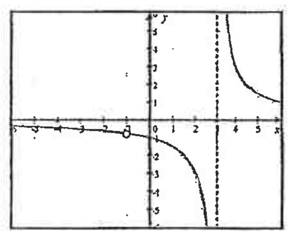


$$y=\frac{-x^2}{x^2-x-6}=\frac{-x^2}{(x-3)(x+2)}$$

Vertical asymptotes. $x-3=0\Rightarrow x=3$ and $x+2=0\Rightarrow x=-2$
Horizontal asymptotes: Since the highest power of x is 2 in both numerator and denominator, there is a horizontal asymptote at $y=-1$

This is confirmed by the graph to the right Note that the curve actually crosses its horizontal asymptote on the left side of the graph.

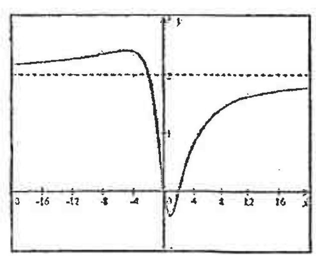
2) Find any vertical and horizontal asymptotes for the graph of $y=\frac{3x+3}{x^2-2x-3}$



$$y=\frac{3x+3}{x^2-2x-3}=\frac{3(x+1)}{(x-3)(x+1)}=\frac{3}{x-3}$$

Vertical asymptotes: $x-3=0\Rightarrow x=3$ Note that since the $(x+1)$ cancels, there is no vertical asymptote at $x = 1$, but a hole (sometimes called a removable discontinuity) in the graph.
Horizontal asymptotes. Since there the highest power of x is in the denominator, there is a horizontal asymptote at $y = 0$ (the x-axis). This is confirmed by the graph to the right.

3) Find any vertical and horizontal asymptotes for the graph of $y=\frac{2x^2-4x}{x^2+4}$



$$y=\frac{2x^2-4x}{x^2+4}=\frac{2x(x-2)}{x^2+4}$$

Vertical asymptotes. None. The denominator doesn't factor and setting it equal to zero has no solutions.
Horizontal asymptotes: Since the highest power of x is 2 in both numerator and denominator, there is a horizontal asymptote at $y = 2$. This is confirmed by the graph to the right.

I. Asymptotes - Assignment

- Find any vertical and horizontal asymptotes and if present, the location of holes, for the graph of

1. $y=\dfrac{x-1}{x+5}$

2. $y=\dfrac{8}{x^2}$

3. $y=\dfrac{2x+16}{x+8}$

4. $y=\dfrac{2x^2+6x}{x^2+5x+6}$

5. $y=\dfrac{x}{x^2-25}$

6. $y=\dfrac{x^2-5}{2x^2-12}$

7. $y=\dfrac{4+3x-x^2}{3x^2}$

8. $y=\dfrac{5x+1}{x^2-x-1}$

9. $y=\dfrac{1-x-5x^2}{x^2+x+1}$

10. $y=\dfrac{x^3}{x^2+4}$

11. $y=\dfrac{x^3+4x}{x^3-2x^2+4x-8}$

12. $y=\dfrac{10x+20}{x^3-2x^2-4x+8}$

13. $y=\dfrac{1}{x}-\dfrac{x}{x+2}$ (hint: express with a common denominator)

J. Negative and Fractional Exponents

In calculus, you will be required to perform algebraic manipulations with **negative exponents** as well as **fractional exponents**. You should know the definition of a negative exponent: $x^{-n} = \frac{1}{x^n}, x \neq 0$ Note that negative powers do not make expressions negative; they create fractions. Typically expressions in multiple-choice answers are written with positive exponents and students are required to eliminate negative exponents. Fractional exponents create roots. The definition of $x^{1/2} = \sqrt{x}$ and $x^{a/b} = \sqrt[b]{x^a} = \left(\sqrt[b]{x}\right)^a$

As a reminder when we multiply, we add exponents. $\left(x^a\right)\left(x^b\right) = x^{a+b}$

When we divide, we subtract exponents: $\frac{x^a}{x^b} = x^{a-b}, x \neq 0$

When we raise powers, we multiply exponents: $\left(x^a\right)^b = x^{ab}$

In your algebra course, leaving an answer with a radical in the denominator was probably not allowed. You had to rationalize the denominator $\frac{1}{\sqrt{x}}$ changed to $\left(\frac{1}{\sqrt{x}}\right)\left(\frac{\sqrt{x}}{\sqrt{x}}\right) = \frac{\sqrt{x}}{x}$ In calculus, you will find that it is not necessary to rationalize and it is recommended that you not take the time to do so.

• Simplify and write with positive exponents. Note: # 12 involves complex fractions, covered in section K.

1 $-8x^{-2}$

$\boxed{\frac{-8}{x^2}}$

2. $\left(-5x^3\right)^{-2}$

$\boxed{(-5)^{-2}x^{-6} = \frac{1}{(-5)^2 x^6} = \frac{1}{25x^6}}$

3 $\left(\frac{-3}{x^4}\right)^{-2}$

$\boxed{\frac{(-3)^{-2}}{\left(x^4\right)^{-2}} = \frac{1}{(-3)^2 x^{-8}} = \frac{x^8}{9}}$

4 $\left(36x^{10}\right)^{1/2}$

$\boxed{6x^5}$

5 $\left(27x^3\right)^{-2/3}$

$\boxed{\frac{1}{\left(27x^3\right)^{2/3}} = \frac{1}{9x^2}}$

6. $\left(16x^{-2}\right)^{3/4}$

$\boxed{16^{3/4}x^{-4/3} = \frac{8}{x^{4/3}}}$

7 $\left(x^{1/2} - x\right)^{-2}$

$\boxed{\frac{1}{\left(x^{1/2} - x\right)^2} = \frac{1}{x - 2x^{3/2} + x^2}}$

8. $\left(4x^2 - 12x + 9\right)^{-1/2}$

$\boxed{\frac{1}{\left[(2x-3)^2\right]^{1/2}} = \frac{1}{2x-3}}$

9 $\left(x^{1/3}\right)\left(\frac{1}{2}x^{-1/2}\right) + \left(x^{1/2} + 1\right)\left(\frac{1}{3}x^{-1/3}\right)$

$\boxed{\frac{x^{1/3}}{2x^{1/2}} + \frac{x^{1/2} + 1}{3x^{1/3}} = \frac{1}{2x^{1/6}} + \frac{x^{1/2} + 1}{3x^{1/3}}}$

10 $\frac{-2}{3}(8x)^{-5/3}(8)$

$\boxed{\frac{-16}{3(8x)^{5/3}} = \frac{-16}{3(32)x^{5/3}} = -\frac{1}{6x^{5/3}}}$

11 $\frac{(x+4)^{1/2}}{(x-4)^{-1/2}}$

$\boxed{(x+4)^{1/2}(x-4)^{1/2} = \left(x^2 - 16\right)^{1/2}}$

12. $\left(x^{-1} + y^{-1}\right)^{-1}$

$\boxed{\left(\frac{1}{\frac{1}{x} + \frac{1}{y}}\right)\left(\frac{xy}{xy}\right) = \frac{xy}{y+x}}$

J. Negative and Fractional Exponents - Assignment

Simplify and write with positive exponents.

1. $-12^2 x^{-5}$

2. $(-12x^5)^{-2}$

3. $(4x^{-1})^{-1}$

4. $\left(\frac{-4}{x^4}\right)^{-3}$

5. $\left(\frac{5x^3}{y^2}\right)^{-3}$

6. $(x^3-1)^{-2}$

7 $(121x^8)^{1/2}$

8. $(8x^2)^{-4/3}$

9. $(-32x^{-5})^{-3/5}$

10. $(x+y)^{-2}$

11. $(x^3+3x^2+3x+1)^{-2/3}$

12. $x(x^{1/2}-x)^{-2}$

13. $\frac{1}{4}(16x^2)^{-3/4}(32x)$

14. $\frac{(x^2-1)^{-1/2}}{(x^2+1)^{1/2}}$

15. $(x^{-2}+2^{-2})^{-1}$

K. Eliminating Complex Fractions

Calculus frequently uses **complex fractions**, which are fractions within fractions. Answers are never left with complex fractions and they must be eliminated. There are two methods to eliminate complex fractions.

When the problem is in the form of $\frac{\frac{a}{b}}{\frac{c}{d}}$, we can "flip the denominator" and write it as $\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$

However, this does not work when the numerator and denominator are not single fractions. The best way is to eliminate the complex fractions in all cases is to find the LCD (lowest common denominator) of all the fractions in the complex fraction. Multiply all terms by this LCD and you are left with a fraction that is magically no longer complex. **Important:** Note that $\frac{x^{-1}}{y^{-1}}$ can be written as $\frac{y}{x}$ but $\frac{1+x^{-1}}{y^{-1}}$ must be written as $\frac{1+\frac{1}{x}}{\frac{1}{y}}$

• Eliminate the complex fractions.

1. $\dfrac{\frac{2}{3}}{\frac{5}{6}}$

$$\left(\frac{\frac{2}{3}}{\frac{5}{6}}\right)\left(\frac{6}{6}\right)=\frac{4}{5}$$

2. $\dfrac{1+\frac{2}{3}}{1+\frac{5}{6}}$

$$\left(\frac{1+\frac{2}{3}}{1+\frac{5}{6}}\right)\left(\frac{6}{6}\right)=\frac{6+4}{6+5}=\frac{10}{11}$$

3. $\dfrac{\frac{3}{4}+\frac{5}{3}}{2-\frac{1}{6}}$

$$\left(\frac{\frac{3}{4}+\frac{5}{3}}{2-\frac{1}{6}}\right)\left(\frac{12}{12}\right)=\frac{9+20}{24-2}=\frac{29}{22}$$

4. $\dfrac{1+\frac{1}{2}x^{-1}}{1+\frac{1}{3}x^{-1}}$

$$\left(\frac{1+\frac{1}{2x}}{1+\frac{1}{3x}}\right)\left(\frac{6x}{6x}\right)=\frac{6x+3}{6x+2}$$

5. $\dfrac{x-\frac{1}{2x}}{x^2+\frac{1}{4x^2}}$

$$\left(\frac{x-\frac{1}{2x}}{x^2+\frac{1}{4x^2}}\right)\left(\frac{4x^2}{4x^2}\right)=\frac{4x^3-2x}{4x^4+1}$$

6. $\dfrac{\frac{2}{5}x^{5/3}}{\frac{5}{3}}$

$$\left(\frac{\frac{2}{5}x^{5/3}}{\frac{5}{3}}\right)\left(\frac{15}{15}\right)=\frac{6x^{5/3}}{25}$$

7. $\dfrac{x^{-3}+x}{x^{-2}+1}$

$$\left(\frac{\frac{1}{x^3}+x}{\frac{1}{x^2}+1}\right)\left(\frac{x^3}{x^3}\right)=\frac{1+x^4}{x+x^3}$$

8. $\dfrac{\frac{1}{2}(2x+5)^{-2/3}}{\frac{-2}{3}}$

$$\left(\frac{\frac{1}{2}}{\frac{-2}{3}(2x+5)^{2/3}}\right)\frac{6}{6}=\frac{-3}{4(2x+5)^{2/3}}$$

9. $\dfrac{(x-1)^{1/2}-\frac{x(x-1)^{-1/2}}{2}}{x-1}$

$$\left(\frac{(x-1)^{1/2}-\frac{x}{2(x-1)^{1/2}}}{x-1}\right)\left[\frac{2(x-1)^{1/2}}{2(x-1)^{1/2}}\right]$$

$$\frac{2(x-1)-x}{2(x-1)^{3/2}}=\frac{x-2}{2(x-1)^{3/2}}$$

K. Eliminating Complex Fractions - Assignment

- Eliminate the complex fractions.

1. $\dfrac{\frac{5}{8}}{\frac{-2}{3}}$

2. $\dfrac{4-\frac{2}{9}}{3+\frac{4}{3}}$

3. $\dfrac{2+\frac{7}{2}+\frac{3}{5}}{5-\frac{3}{4}}$

4. $\dfrac{x-\frac{1}{x}}{x+\frac{1}{x}}$

5. $\dfrac{1+x^{-1}}{1-x^{-2}}$

6. $\dfrac{x^{-1}+y^{-1}}{x+y}$

7. $\dfrac{x^{-2}+x^{-1}+1}{x^{-2}-x}$

8. $\dfrac{\frac{1}{3}(3x-4)^{-3/4}}{\frac{-3}{4}}$

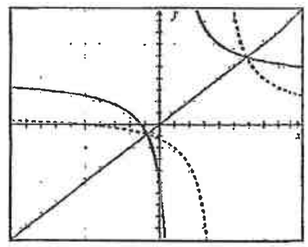
9. $\dfrac{2x(2x-1)^{1/2}-2x^{2}(2x-1)^{-1/2}}{(2x-1)}$

L. Inverses

No topic in math confuses students more than inverses. If a function is a rule that maps x to y, an **inverse** is a rule that brings y back to the original x. If a point (x,y) is a point on a function f, then the point (y,x) is on the inverse function f^{-1} Students mistakenly believe that since $x^{-1}=\frac{1}{x}$, then $f^{-1}=\frac{1}{f}$ This is decidedly incorrect.
If a function is given in equation form, to find the inverse, replace all occurrences of x with y and all occurrences of y with x If possible, then solve for y Using the "horizontal-line test" on the original function f will quickly determine whether or not f^{-1} is also a function. By definition, $f\left(f^{-1}(x)\right)=x$ The domain of f^{-1} is the range of f and the range of f^{-1} is the domain of f

1 Find the inverse to $y=\frac{4x+5}{x-1}$ and show graphically that its inverse is a function.

Inverse: $x=\frac{4y+5}{y-1}\Rightarrow xy-x=4y+5\Rightarrow y=\frac{x+5}{x-4}$



Note that the function is drawn in bold and the inverse as dashed. The function and its inverse is symmetrical to the line $y=x$. The inverse is a function for two reasons. a) it passes the vertical line text or b) the function passes the horizontal line test.

2. Find the inverse to the following functions and show graphically that its inverse is a function.

a. $y=4x-3$

Inverse: $x=4y-3$
$y=\frac{x+3}{4}$ (function)

b. $y=x^2+1$

Inverse: $x=y^2+1$
$y=\pm\sqrt{x-1}$ (not a function)

c. $y=x^2+4x+4$

Inverse: $x=y^2+4y+4$
$x=(y+2)^2\Rightarrow\pm\sqrt{x}=y+2$
$y=-2\pm\sqrt{x}$ (not a function)

3. Find the inverse to the following functions and show that $f\left(f^{-1}(x)\right)=x$

a. $f(x)=7x+4$

Inverse: $x=7y+4$
$y=f^{-1}(x)=\frac{x-4}{7}$
$f\left(\frac{x-4}{7}\right)=7\left(\frac{x-4}{7}\right)+4=x$

b. $f(x)=\frac{1}{x-1}$

Inverse: $x=\frac{1}{y-1}\Rightarrow xy-x=1$
$y=f^{-1}(x)=\frac{x+1}{x}$
$f\left(\frac{x+1}{x}\right)=\left(\frac{1}{\frac{x+1}{x}-1}\right)\left(\frac{x}{x}\right)$
$=\frac{x}{x+1-x}=x$

c. $f(x)=x^3-1$

Inverse: $x=y^3-1$
$y=f^{-1}(x)=\sqrt[3]{x+1}$
$f\left(\sqrt[3]{x+1}\right)=\left(\sqrt[3]{x+1}\right)^3-1=x$

4 Without finding the inverse, find the domain and range of the inverse to $f(x)=\sqrt{x-2}+3$

Function. Domain: $[-2,\infty)$, Range. $[3,\infty)$ Inverse: Domain. $[3,\infty)$, Range: $[-2,\infty)$

L. Inverses – Assignment

1 Find the inverse to the following functions and show graphically that its inverse is a function.

a. $2x-6y=1$

b. $y=ax+b$

c. $y=9-x^2$

d. $y=\sqrt{1-x^3}$

e. $y=\frac{9}{x}$

f. $y=\frac{2x+1}{3-2x}$

2. Find the inverse to the following functions and show that $f\left(f^{-1}(x)\right)=x$

a. $f(x)=\frac{1}{2}x-\frac{4}{5}$

b $f(x)=x^2-4$

c. $f(x)=\frac{x^2}{x^2+1}$

3. Without finding the inverse, find the domain and range of the inverse to $f(x)=\frac{\sqrt{x+1}}{x^2}$

M. Adding Fractions and Solving Fractional Equations

There are two major problem types with fractions. Adding/subtracting fractions and solving fractional equations. Algebra has taught you that in order to add fractions, you need to find an LCD and *multiply each fraction by one* in such a way that you obtain the LCD in each fraction. However, when you solve fractional equations (equations that involve fractions), you still find the LCD but you *multiply every term by the LCD* When you do that, all the fractions disappear, leaving you with an equation that is hopefully solvable. Answers should be checked in the original equation.

1 a. Combine: $\frac{x}{3}-\frac{x}{4}$

LCD· 12 $\quad \frac{x}{3}\left(\frac{4}{4}\right)-\frac{x}{4}\left(\frac{3}{3}\right)$

$\frac{4x-3x}{12}=\frac{x}{12}$

b. Solve: $\frac{x}{3}-\frac{x}{4}=12$

$12\left(\frac{x}{3}\right)-12\left(\frac{x}{4}\right)=12(12)$

$4x-3x=144 \Rightarrow x=144$

$x=144 \quad \frac{144}{3}-\frac{144}{4}=48-36=12$

2. a. Combine $x+\frac{6}{x}$

LCD· x $\quad x\left(\frac{x}{x}\right)+\frac{6}{x}$

$\frac{x^2+6}{x}$

b. Solve. $x+\frac{6}{x}=5$

$x(x)+x\left(\frac{6}{x}\right)=5x$

$x^2+6=5x \Rightarrow x^2-5x+6=0$

$(x-2)(x-3)=0 \Rightarrow x=2,\ x=3$

$x=2 \quad 2+\frac{6}{2}=2+3=5 \quad x=3 \quad 3+\frac{6}{3}=3+2=5$

3 a. Combine: $\frac{12}{x+2}-\frac{4}{x}$

LCD· $x(x+2)$ $\quad \left(\frac{12}{x+2}\right)\left(\frac{x}{x}\right)-\frac{4}{x}\left(\frac{x+2}{x+2}\right)$

$\frac{12x-4x-8}{x(x+2)}$

$\frac{8x-8}{x(x+2)}$

b. Solve $\frac{12}{x+2}-\frac{4}{x}=1$

$\frac{12}{x+2}(x)(x+2)-\frac{4}{x}(x)(x+2)=1(x)(x+2)$

$12x-4x-8=x^2+2x \Rightarrow x^2-6x+8=0$

$(x-2)(x-4)=0 \Rightarrow x=2,4$

$x=2 \quad \frac{12}{4}-\frac{4}{2}=3-2=1 \quad x=4 \quad \frac{12}{6}-\frac{4}{4}=2-1=1$

4 a. $\frac{x}{2x-6}-\frac{3}{x^2-6x+9}$

LCD· $2(x-3)^2$

$\frac{x}{2(x-3)}\left(\frac{x-3}{x-3}\right)-\frac{3}{(x-3)^2}\left(\frac{2}{2}\right)$

$\frac{x^2-3x-6}{2(x-3)^2}$

b. Solve $\frac{x}{2x-6}-\frac{3}{x^2-6x+9}=\frac{x-2}{3x-9}$

$\left[\frac{x}{2(x-3)}-\frac{3}{(x-3)^2}=\frac{x-2}{3(x-3)}\right]6(x-3)^2$

$3x(x-3)-18=2(x-3)(x-2)$

$3x^2-9x-18=2x^2-10x+12$

$x^2+x-30=0 \Rightarrow (x+6)(x-5)=0 \Rightarrow x=-6,5$

$x=-6 \quad \frac{-6}{-18}-\frac{3}{81}=\frac{-8}{-27} \quad x=5 \quad \frac{5}{4}-\frac{3}{4}=\frac{3}{6}$

M. Adding Fractions and Solving Fractional Equations - Assignment

1 a. Combine. $\frac{2}{3}-\frac{1}{x}$

b Solve: $\frac{2}{3}-\frac{1}{x}=\frac{5}{6}$

2. a. Combine: $\frac{1}{x-3}+\frac{1}{x+3}$

b Solve: $\frac{1}{x-3}+\frac{1}{x+3}=\frac{10}{x^2-9}$

3 a. Combine: $\frac{5}{2x}-\frac{5}{3x+15}$

b. Solve: $\frac{5}{2x}-\frac{5}{3(x+5)}=\frac{5}{x}$

4 a. Combine: $\frac{2x-1}{x-1}-\frac{3x}{2x+1}$

b Solve: $\frac{2x-1}{x-1}-\frac{3x}{2x+1}=\frac{x^2+11}{2x^2-x-1}$

N. Solving Absolute Value Equations

Absolute value equations crop up in calculus, especially in BC calculus. The definition of the absolute value function is a piecewise function. $f(x)=|x|=\begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$ So, to solve an absolute value equation, split the absolute value equation into two equations, one with a positive parentheses and the other with a negative parentheses and solve each equation. It is possible that this procedure can lead to incorrect solutions so solutions must be checked.

- Solve the following equations.

1 $|x-1|=3$

$x-1=3$	$-(x-1)=3$
$x=4$	$-x+1=3$
	$x=-2$

2. $|3x+2|=9$

$3x+2=9$	$-(3x+2)=9$
$3x=7$	$-3x-2=9$
$x=\frac{7}{3}$	$3x=-11$
	$x=\frac{-11}{3}$

3. $|2x-1|-x=5$

$2x-1-x=5$	$-(2x-1)-x=5$
$x=6$	$-3x=4$
	$x=\frac{-4}{3}$

4 $|x+5|+5=0$

$x+5+5=0$	$-(x+5)+5=0$
$x=-10$	$-x-5+5=0$
	$x=0$

Both answers are invalid. It is impossible to add 5 to an absolute value and get 0

5 $|x^2-x|=2$

$(x^2-x)=2$	$-(x^2-x)=2$
$x^2-x-2=0$	$-x^2+x=2$
$(x-2)(x+1)=0$	$0=x^2+x+2$
$x=2, x=-1$	No real solution

Both solutions check

6 $|x-10|=x^2-10x$

$x-10=x^2-10x$	$-(x-10)=x^2-10x$
$x^2-11x+10=0$	$-x+10=x^2-10x$
$(x-1)(x-10)=0$	$x^2-9x-10=0$
$x=1, x=10$	$(x-10)(x+1)=0$
	$x=10, x=-1$

Of the three solutions, only $x=-1$ and $x=10$ are valid.

7 $|x|+|2x-2|=8$

$x+2x-2=8$	$-x+2x-2=8$	$x-(2x-2)=8$	$-x-(2x-2)=8$
$3x=10$	$x=10$	$-x=6$	$-3x=6$
$x=\frac{10}{3}$		$x=-6$	$x=-2$

Of the four solutions, only $x=\frac{10}{3}$ and $x=-2$ are valid

N. Solving Absolute Value Equations - Assignment

- Solve the following equations.

1. $4|x+8|=20$

2. $|1-7x|=13$

3. $|8+2x|+2x=40$

4. $|4x-5|+5x+2=0$

5. $|x^2-2x-1|=7$

6. $|12-x|=x^2-12x$

7. $|x|+|4x-4|+x=14$

O. Solving Inequalities

You may think that solving inequalities are just a matter of replacing the equal sign with an inequality sign. In reality, they can be more difficult and are fraught with dangers. And in calculus, inequalities show up more frequently than solving equations. Solving inequalities are a simple matter if they are based on linear equations. They are solved exactly like linear equations, remembering that if you multiply or divide both sides by a negative number, the direction of the inequality sign must be reversed.

However, if the inequality is more complex than a linear function, it is advised to bring all terms to one side. Pretend for a moment it is an equation and solve. Then create a number line which determines whether the transformed inequality is positive or negative in the intervals created on the number line and choose the correct intervals according to the inequality, paying attention to whether the zeroes are included or not.

If the inequality involves an absolute value, create two equations, replacing the absolute value with a positive parentheses and a negative parentheses and the inequality sign with an equal sign. Solve each, placing each solution on your number line. Then determine which intervals satisfy the original inequality

If the inequality involves a rational function, set both numerator and denominator equal to zero, which will give you the values you need for your number line. Determine whether the inequality is positive or negative in the intervals created on the number line and choose the correct intervals according to the inequality, paying attention to whether the endpoints are included or not.

• Solve the following inequalities.

1 $2x-8\le 6x+2$

$-10\le 4x$ or $-4x\le 10$

$\dfrac{-5}{2}\le x$ or $x\ge\dfrac{-5}{2}$

2. $1-\dfrac{3x}{2}>x-5$

$2-3x>2x-10$

$12>5x\Rightarrow x<\dfrac{12}{5}$

3 $-5\le 6x-1<11$

$-6\le 6x\le 12$

$-1\le x\le 2$

4. $|2x-1|\le x+4$

$|2x-1|-x-4\le 0$

$2x-1-x-4=0$ $\quad -2x+1-x-4=0$

$x=5$ $\quad x=-1$

++++++0----------0+++++++

1 $\quad$ 5

So $-1\le x\le 5$ or $[-1,5]$

5 $x^2-3x>18$

$x^2-3x-18>0\Rightarrow (x+3)(x-6)>0$

For $(x+3)(x-6)=0$, $x=-3, x=6$

++++++0-------0+++++++

-3 $\quad$ 6

So $x<-3$ or $x>6$ or $(-\infty,-3)\cup(6,\infty)$

6. $\dfrac{2x-7}{x-5}\le 1$

$\dfrac{2x-7}{x-5}-1=0\Rightarrow\dfrac{2x-7}{x-5}-\dfrac{x-5}{x-5}<0\Rightarrow\dfrac{x-2}{x-5}<0$

++++++0-------∞+++++++

2 $\quad$ 5

So $2\le x<5$ or $[2,5)$

7 Find the domain of $\sqrt{32-2x^2}$

$2(4+x)(4-x)\ge 0$

------0+++++++0-------

-4 $\quad$ 4

So $-4\le x\le -4$ or $[-4,4]$

O. Solving Inequalities - Assignment

• Solve the following inequalities.

1. $5(x-3) \le 8(x+5)$

2. $4-\frac{5x}{3} > -\left(2x+\frac{1}{2}\right)$

3 $\frac{3}{4} > x+1 > \frac{1}{2}$

4. $x+7 \ge |5-3x|$

5. $(x+2)^2 < 25$

6. $x^3 < 4x^2$

7. $\frac{5}{x-6} \ge \frac{1}{x+2}$

8. Find the domain of $\sqrt{\frac{x^2-x-6}{x-4}}$

P. Exponential Functions and Logarithms

Calculus spends a great deal of time on exponential functions in the form of b^x Don't expect that when you start working with them in calculus, your teacher will review them. So learn them now! Students must know that the definition of a **logarithm** is based on exponential equations. If $y = b^x$ then $x = \log_b y$ So when you are trying to find the value of $\log_2 32$, state that $\log_2 32 = x$ and $2^x = 32$ and therefore $x = 5$

If the base of a log statement is not specified, it is defined to be 10 When we asked for log 100, we are solving the equation. $10^x = 100$ and $x = 2$. The function $y = \log x$ has domain $(0,\infty)$ and range $(-\infty,\infty)$ In calculus, we primarily use logs with base *e*, which are called natural logs (ln). So finding ln 5 is the same as solving the equation $e^x = 5$ Students should know that the value of $e = 2.71828$.

There are three rules that students must keep in mind that will simplify problems involving logs and natural logs. These rules work with logs of any base including natural logs.

i. $\log a + \log b = \log(a\ b)$ ii. $\log a - \log b = \log\left(\frac{a}{b}\right)$ iii. $\log a^b = b\log a$

1 Find a. $\log_4 8$

$\log_4 8 = x$
$4^x = 8 \Rightarrow 2^{2x} = 2^3$
$x = \frac{3}{2}$

b. $\ln\sqrt{e}$

$\ln\sqrt{e} = x$
$e^x = e^{1/2}$
$x = \frac{1}{2}$

c. $10^{\log 4}$

$\log 4 = x$
$10^x = 4$ so $10^{\log 4} = 4$
10 to a power and log are inverses

d. $\log 2 + \log 50$

$\log(2\ 50) = \log 100$
2

e. $\log_4 192 - \log_4 3$

$\log_4\left(\frac{192}{3}\right)$
$\log_4 64 = 3$

f. $\ln\sqrt[5]{e^3}$

$\ln e^{3/5} = \frac{3}{5}\ln e = \frac{3}{5}$

2. Solve a. $\log_9\left(x^2 - x + 3\right) = \frac{1}{2}$

$x^2 - x + 3 = 9^{1/2}$
$x(x-1) = 0$
$x = 0, x = 1$

b. $\log_{36} x + \log_{36}(x-1) = \frac{1}{2}$

$\log_{36} x(x-1) = \frac{1}{2}$
$x(x-1) = 36^{1/2} = 6$
$x^2 - x - 6 = 0$
$(x-3)(x+2) = 0$
Only $x = 3$ is in the domain

c. $\ln x - \ln(x-1) = 1$

$\ln\left(\frac{x}{x-1}\right) = 1$
$\frac{x}{x-1} = e \Rightarrow x = ex - e$
$x = \frac{e}{e-1}$

d. $5^x = 20$

$\log(5^x) = \log 20$
$x\log 5 = \log 20$
$x = \frac{\log 20}{\log 5}$ or $x = \frac{\ln 20}{\ln 5}$

e. $e^{-2x} = 5$

$\ln e^{-2x} = \ln 5$
$-2x = \ln 5 \Rightarrow x = \frac{-\ln 5}{2}$

f. $2^x = 3^{x-1}$

$\log(2^x) = \log(3^{x-1})$
$x\log 2 = (x-1)\log 3$
$x\log 2 = x\log 3 - \log 3 \Rightarrow x = \frac{\log 3}{\log 3 - \log 2}$

P. Exponential Functions and Logarithms - Assignment

1 Find a. $\log_2 \frac{1}{4}$

b $\log_8 4$

c. $\ln \frac{1}{\sqrt[3]{e^2}}$

d. $5^{\log_5 40}$

e. $e^{\ln 12}$

f. $\log_{12} 2 + \log_{12} 9 + \log_{12} 8$

g. $\log_2 \frac{2}{3} + \log_2 \frac{3}{32}$

h. $\log_{\frac{1}{3}} \frac{4}{3} - \log_{\frac{1}{3}} 12$

i. $\log_3 \left(\sqrt{3}\right)^5$

2. Solve a. $\log_5(3x-8)=2$

b. $\log_9\left(x^2-x+3\right)=\frac{1}{2}$

c. $\log(x-3)+\log 5=2$

d. $\log_2(x-1)+\log_2(x+3)=5$

e. $\log_5(x+3)-\log_5 x=2$

f. $\ln x^3-\ln x^2=\frac{1}{2}$

g. $3^{x-2}=18$

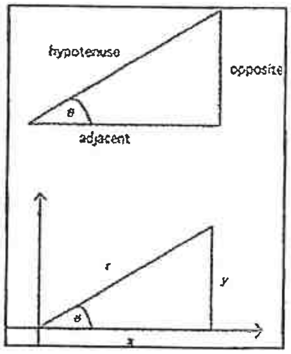
h. $e^{3x+1}=10$

i. $8^x=5^{2x-1}$

Q. Right Angle Trigonometry

Trigonometry is an integral part of AP calculus. Students must know the basic trig function definitions in terms of opposite, adjacent and hypotenuse as well as the definitions if the angle is in standard position.

Given a right triangle with one of the angles named θ, and the sides of the triangle relative to θ named opposite (y), adjacent (x), and hypotenuse (r) we define the 6 trig functions to be:



$\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{y}{r}$ $\qquad$ $\csc\theta = \dfrac{\text{hyptotenuse}}{\text{opposite}} = \dfrac{r}{y}$

$\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{x}{r}$ $\qquad$ $\sec\theta = \dfrac{\text{hyptotenuse}}{\text{adjacent}} = \dfrac{r}{x}$

$\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{y}{x}$ $\qquad$ $\cot\theta = \dfrac{\text{adjacent}}{\text{opposite}} = \dfrac{x}{y}$

The Pythagorean theorem ties these variables together: $x^2 + y^2 = r^2$ Students should recognize right triangles with integer sides. 3-4-5, 5-12-13, 8-15-17, 7-24-25 Also any multiples of these sides are also sides of a right triangle. Since r is the largest side of a right triangle, it can be shown that the range of $\sin\theta$ and $\cos\theta$ is $[-1,1]$, the range of $\csc\theta$ and $\sec\theta$ is $(-\infty,-1]\cup[1,\infty)$ and the range of $\tan\theta$ and $\cot\theta$ is $(-\infty,\infty)$.

Also vital to master is the signs of the trig functions in the four quadrants. A good way to remember this is A – S – T – C where All trig functions are positive in the 1st quadrant, Sin is positive in the 2nd quadrant, Tan is positive in the 3rd quadrant and Cos is positive in the 4th quadrant.

1 Let be a point on the terminal side of θ Find the 6 trig functions of θ (Answers need not be rationalized).

a) $P(-8,6)$

$x=-8, y=6, r=10$

$\sin\theta = \dfrac{3}{5}$ $\qquad$ $\csc\theta = \dfrac{5}{3}$

$\cos\theta = -\dfrac{4}{5}$ $\qquad$ $\sec\theta = -\dfrac{5}{4}$

$\tan\theta = -\dfrac{3}{4}$ $\qquad$ $\cot\theta = -\dfrac{4}{3}$

b $P(1,3)$

$x=1,\ y=3,\ r=\sqrt{10}$

$\sin\theta = \dfrac{3}{\sqrt{10}}$ $\qquad$ $\csc\theta = \dfrac{\sqrt{10}}{3}$

$\cos\theta = \dfrac{1}{\sqrt{10}}$ $\qquad$ $\sec\theta = \sqrt{10}$

$\tan\theta = 3$ $\qquad$ $\cot\theta = \dfrac{1}{3}$

c. $P\left(-\sqrt{10},-\sqrt{6}\right)$

$x=-\sqrt{10},\ y=-\sqrt{6},\ r=4$

$\sin\theta = -\dfrac{\sqrt{6}}{4}$ $\qquad$ $\csc\theta = -\dfrac{4}{\sqrt{6}}$

$\cos\theta = -\dfrac{\sqrt{10}}{4}$ $\qquad$ $\sec\theta = -\dfrac{4}{\sqrt{10}}$

$\tan\theta = \sqrt{\dfrac{3}{5}}$ $\qquad$ $\cot\theta = \sqrt{\dfrac{5}{3}}$

2. If $\cos\theta = \dfrac{2}{3}$, θ in quadrant IV, find $\sin\theta$ and $\tan\theta$

$x=2,\ r=3,\ y=-\sqrt{5}$

$\sin\theta = -\dfrac{\sqrt{5}}{3}, \tan\theta = -\dfrac{\sqrt{5}}{2}$

3 If $\sec\theta = \sqrt{3}$ find $\sin\theta$ and $\tan\theta$

θ is in quadrant I or IV

$x=1,\ y=\pm\sqrt{2},\ r=\sqrt{3}$

$\sin\theta = \pm\sqrt{\dfrac{2}{3}},\ \tan\theta = \pm\sqrt{2}$

4 Is $3\cos\theta + 4 = 2$ possible?

$3\cos\theta = -2$

$\cos\theta = -\dfrac{2}{3}$ which is possible.

Q. Right Angle Trigonometry - Assignment

1. Let be a point on the terminal side of θ Find the 6 trig functions of θ (Answers need not be rationalized).

a) $P(15,8)$

b. $P(-2,3)$

c. $P\left(-2\sqrt{5},-\sqrt{5}\right)$

2. If $\tan\theta=\frac{12}{5}$, θ in quadrant III, find $\sin\theta$ and $\cos\theta$

3. If $\csc\theta=\frac{6}{5}$, θ in quadrant II, find $\cos\theta$ and $\tan\theta$

4 $\cot\theta=\frac{-2\sqrt{10}}{3}$ find $\sin\theta$ and $\cos\theta$

5. Find the quadrants where the following is true: Explain your reasoning.

a. $\sin\theta>0$ and $\cos\theta<0$

b $\csc\theta<0$ and $\cot\theta>0$

c. all functions are negative

6. Which of the following is possible? Explain your reasoning.

a. $5\sin\theta=-2$

b. $3\sin\alpha+4\cos\beta=8$

c. $8\tan\theta+22=85$

R. Special Angles

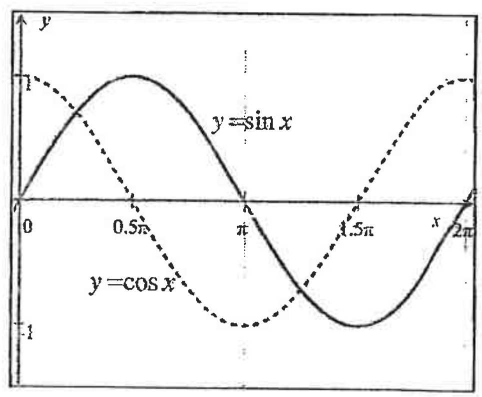
Students must be able to find trig functions of quadrant angles $(0,\ 90°, 180°, 270°)$ and special angles, those based on the $30°-60°-90°$ and $45°-45°-90°$ triangles.

First, for most calculus problems, angles are given and found in radians. Students must know how to convert degrees to radians and vice-versa. The relationship is 2π radians $= 360°$ or π radians $= 180°$ Angles are assumed to be in radians so when an angle of $\frac{\pi}{3}$ is given, it is in radians. However, a student should be able to picture this angle as $\frac{180°}{3} = 60°$ It may be easier to think of angles in degrees than radians, but realize that unless specified, angle measurement must be written in radians. For instance, $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

The trig functions of **quadrant angles** $\left(0,\ 90°, 180°, 270° \text{ or } 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right)$ can quickly be found. Choose a point along the angle and realize that r is the distance from the origin to that point and always positive. Then use the definitions of the trig functions.

θ	point	x	y	r	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0	$(1,0)$	1	0	1	0	1	0	does not exist	1	does not exist
$\frac{\pi}{2}$ or $90°$	$(0,1)$	0	1	1	1	0	does not exist	1	does not exist	0
π or $180°$	$(-1,0)$	-1	0	1	0	-1	0	does not exist	1	does not exist
$\frac{3\pi}{2}$ or $270°$	$(0,-1)$	0	-1	1	-1	0	Does not exist	-1	does not exist	0

If you picture the graphs of $y = \sin x$ and $y = \cos x$ as shown to the right, you need not memorize the table. You must know these graphs backwards and forwards.



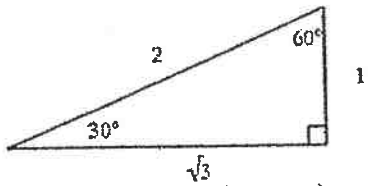
- Without looking at the table, find the value of

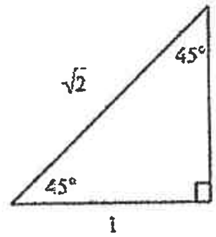
a. $5\cos 180° - 4\sin 270°$

$$\boxed{\begin{array}{l} 5(-1)-4(-1) \\ -5+4=-1 \end{array}}$$

b $\left(\dfrac{8\sin\frac{\pi}{2} - 6\tan\pi}{5\sec\pi - \csc\frac{3\pi}{2}}\right)^2$ $\boxed{\left[\dfrac{8(1)-6(0)}{5(-1)-(-1)}\right]^2 = \left(\dfrac{8}{-4}\right)^2 = 4}$

Because over half of the AP exam does not use a calculator, you must be able to determine trig functions of **special angles**. You must know the relationship of sides in both $30°-60°-90°$ $\left(\frac{\pi}{6},\frac{\pi}{3},\frac{\pi}{2}\right)$ and $45°-45°-90°$ $\left(\frac{\pi}{4},\frac{\pi}{4},\frac{\pi}{2}\right)$ triangles.





In a $30°-60°-90°$ $\left(\frac{\pi}{6},\frac{\pi}{3},\frac{\pi}{2}\right)$ triangle, the ratio of sides is $1-\sqrt{3}-2$.

In a $45°-45°-90°$ $\left(\frac{\pi}{4},\frac{\pi}{4},\frac{\pi}{2}\right)$ triangle, the ratio of sides is $1-1-\sqrt{2}$

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
$30°$ or $\left(\frac{\pi}{6}\right)$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$45°$ or $\left(\frac{\pi}{4}\right)$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60°$ or $\left(\frac{\pi}{3}\right)$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Special angles are any multiple of $30°\left(\frac{\pi}{6}\right)$ or $45°\left(\frac{\pi}{2}\right)$ To find trig functions of any of these angles, draw them and find the **reference angle** (the angle created with the x-axis). Although most problems in calculus will use radians, you might think easier using degrees. This will create one of the triangles above and trig functions can be found, remembering to include the sign based on the quadrant of the angle. Finally, if an angle is outside the range of $0°$ to $360°(0 \text{ to } 2\pi)$, you can always add or subtract $360°(2\pi)$ to find trig functions of that angle. These angles are called **co-terminal angles**. It should be pointed out that $390° \neq 30°$ but $\sin 390° = \sin 30°$

- Find the exact value of the following

a. $4\sin 120° - 8\cos 570°$

> Subtract 360° from 570°
> $4\sin 120° - 8\cos 210°$
> 120° is in quadrant II with reference angle 60°
> 210° is in quadrant III with reference angle 30°
> $4\left(\frac{\sqrt{3}}{2}\right)-8\left(\frac{-\sqrt{3}}{2}\right)=6\sqrt{3}$

b $\left(2\cos\pi - 5\tan\frac{7\pi}{4}\right)^2$

> $(2\cos 180° - 5\tan 315°)^2$
> 180° is a quadrant angle
> 315° is in quadrant III with reference angle 45°
> $[2(-1)-5(-1)]^2 = 9$

R. Special Angles – Assignment

• Evaluate each of the following without looking at a chart.

1. $\sin^2 120° + \cos^2 120°$

2. $2\tan^2 300° + 3\sin^2 150° - \cos^2 180°$

3. $\cot^2 135° - \sin 210° + 5\cos^2 225$

4. $\cot(-30°) + \tan(600°) - \csc(-450°)$

5. $\left(\cos\frac{2\pi}{3} - \tan\frac{3\pi}{4}\right)^2$

6. $\left(\sin\frac{11\pi}{6} - \tan\frac{5\pi}{6}\right)\left(\sin\frac{11\pi}{6} + \tan\frac{5\pi}{6}\right)$

• Determine whether each of the following statements are true or false.

7 $\sin\frac{\pi}{6} + \sin\frac{\pi}{3} = \sin\left(\frac{\pi}{6} + \frac{\pi}{3}\right)$

8 $\dfrac{\cos\frac{5\pi}{3} + 1}{\tan^2\frac{5\pi}{3}} = \dfrac{\cos\frac{5\pi}{3}}{\sec\frac{5\pi}{3} - 1}$

9. $2\left(\frac{3\pi}{2} + \sin\frac{3\pi}{2}\right)\left(1 + \cos\frac{3\pi}{2}\right) > 0$

10. $\dfrac{\cos^3\frac{4\pi}{3} + \sin\frac{4\pi}{3}}{\cos^2\frac{4\pi}{3}} > 0$

S. Trigonometric Identities

Trig identities are equalities involving trig functions that are true for all values of the occurring angles. While you are not asked these identities specifically in calculus, knowing them can make some problems easier The following chart gives the major trig identities that you should know To prove trig identities, you usually start with the more involved expression and use algebraic rules and the fundamental trig identities. A good technique is to change all trig functions to sines and cosines.

Fundamental Trig Identities

$$\csc x = \frac{1}{\sin x},\quad \sec x = \frac{1}{\cos x},\quad \cot x = \frac{1}{\tan x},\quad \tan x = \frac{\sin x}{\cos x},\quad \cot x = \frac{\cos x}{\sin x}$$

$$\sin^2 x + \cos^2 x = 1,\quad 1+\tan^2 x = \sec^2 x,\quad 1+\cot^2 x = \csc^2 x$$

Sum Identities

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \qquad \cos(A+B) = \cos A\cos B - \sin A\sin B$$

Double Angle Identities

$$\sin(2x) = 2\sin x\cos x \qquad \cos(2x) = \cos^2 x - \sin^2 x = 1-2\sin^2 x = 2\cos^2 x - 1$$

- Verify the following identities.

1 $\left(\tan^2 x+1\right)\left(\cos^2-1\right) = -\tan^2 x$

$$\left(\sec^2 x\right)\left(-\sin^2 x\right)$$
$$\left(\frac{1}{\cos^2 x}\right)\left(-\sin^2 x\right)$$
$$-\tan^2 x$$

2. $\sec x - \cos x = \sin x\tan x$

$$\frac{1}{\cos x} - \cos x\left(\frac{\cos x}{\cos x}\right)$$
$$\frac{1-\cos^2 x}{\cos x} = \frac{\sin^2 x}{\cos x}$$
$$\sin x\left(\frac{\sin x}{\cos x}\right) = \sin x\tan x$$

3 $\dfrac{\cot^2 x}{1+\csc x} = \dfrac{1-\sin x}{\sin x}$

$$\left(\frac{\frac{\cos^2}{\sin^2 x}}{1+\frac{1}{\sin x}}\right)\frac{\sin^2 x}{\sin^2 x} = \frac{\cos^2 x}{\sin^2 x+\sin x}$$
$$\frac{1-\sin^2 x}{\sin x(1+\sin x)} = \frac{(1+\sin x)(1-\sin x)}{\sin x(1+\sin x)}$$
$$\frac{1-\sin x}{\sin x}$$

4. $\dfrac{1+\sin x}{\cos x} + \dfrac{\cos x}{1+\sin x} = 2\sec x$

$$\left(\frac{1+\sin x}{\cos x}\right)\left(\frac{1+\sin x}{1+\sin x}\right) + \left(\frac{\cos x}{1+\sin x}\right)\left(\frac{\cos x}{\cos x}\right)$$
$$\frac{1+2\sin x+\sin^2+\cos^2 x}{\cos x(1+\sin x)}$$
$$\frac{1+2\sin x+1}{\cos x(1+\sin x)} = \frac{2+2\sin x}{\cos x(1+\sin x)}$$
$$\frac{2(1+\sin x)}{\cos x(1+\sin x)} = 2\sec x$$

5 $\cos^4 2x - \sin^4 2x = \cos 4x$

$$\left(\cos^2 2x+\sin^2 2x\right)\left(\cos^2 2x-\sin^2 2x\right)$$
$$1\left[\cos 2(2x)\right]$$
$$\cos 4x$$

6. $\sin(3\pi - x) = \sin x$

$$\sin 3\pi\cos x - \cos 3\pi\sin x$$
$$0(\cos x)-(-1)\sin x = \sin x$$

S. Trig Identities – Assignment

• Verify the following identities.

1. $(1+\sin x)(1-\sin x)=\cos^2 x$

2. $\sec^2 x+3=\tan^2 x+4$

3. $\dfrac{1-\sec x}{1-\cos x}=-\sec x$

4. $\dfrac{1}{1+\tan x}+\dfrac{1}{1+\cot x}=1$

5. $\dfrac{\cos x-\cos y}{\sin x+\sin y}+\dfrac{\sin x-\sin y}{\cos x+\cos y}=0$

6. $\dfrac{\sin^3 x+\cos^3 x}{\sin x+\cos x}=1-\sin x\cos x$

7. $\csc 2x=\dfrac{\csc x}{2\cos x}$

8. $\dfrac{\cos 3x}{\cos x}=1-4\sin^2 x$

T. Solving Trig Equations and Inequalities

Trig equations are equations using trig functions. Typically they have many (or infinite) number of solutions so usually they are solved within a specific domain. Without calculators, answers are either quadrant angles or special angles, and again, they must be expressed in radians.

For trig inequalities, set both numerator and denominator equal to zero and solve. Make a sign chart with all these values included and examine the sign of the expression in the intervals. Basic knowledge of the sine and cosine curve is invaluable from section R is invaluable.

• Solve for x on $[0,2\pi)$

1 $x\cos x = 3\cos x$

Do not divide by $\cos x$ as you will lose solutions

$\cos x(x-3)=0$

$\cos x = 0 \qquad x-3=0$

$x=\frac{\pi}{2},\frac{3\pi}{2} \qquad x=3$

You must work in radians.

Saying $x=90°$ makes no sense.

2. $\tan x + \sin^2 x = 2 - \cos^2 x$

$\tan x + \sin^2 x + \cos^2 x = 2$

$\tan x + 1 = 2$

$\tan x = 1$

$x=\frac{\pi}{4},\frac{5\pi}{4}$

Two answers as tangent is positive in quadrants I and III.

3 $3\tan^2 x - 1 = 0$

$3\tan^2 x = 1$

$\tan^2 x = \frac{1}{3}$

$\tan x = \pm\sqrt{\frac{1}{3}} = \pm\frac{\sqrt{3}}{3}$

$x=\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}$

4. $3\cos x = 2\sin^2 x$

$3\cos x = 2(1-\cos^2 x)$

$2\cos^2 x + 3\cos x - 2 = 0$

$(2\cos x - 1)(\cos x + 2) = 0$

$2\cos x = 1 \qquad \cos x = -2$

$\cos x = \frac{1}{2} \qquad$ No solution

$x=\frac{\pi}{3},\frac{5\pi}{3}$

7 Solve for x on $[0,2\pi)$ $\frac{2\cos x+1}{\sin^2 x} > 0$

$2\cos x = -1 \Rightarrow \cos x = \frac{-1}{2} \Rightarrow x = \frac{2\pi}{3},\frac{4\pi}{3}$

$\sin^2 x = 0 \Rightarrow x = 0, \pi$

$++++++ +0 - - - - - - - \infty - - - - - - - - 0 + + + + + + +$

$0 \qquad \frac{2\pi}{3} \qquad \pi \qquad \frac{4\pi}{3} \qquad 2\pi$

Answer: $\left[0,\frac{2\pi}{3}\right)\cup\left(\frac{4\pi}{3},2\pi\right)$

T. Solving Trig Equations and Inequalities - Assignment

- Solve for x on $[0,2\pi)$

1. $\sin^2 x = \sin x$

2. $3\tan^3 x = \tan x$

3. $\sin^2 x = 3\cos^2 x$

4. $\cos x + \sin x \tan x = 2$

5. $\sin x = \cos x$

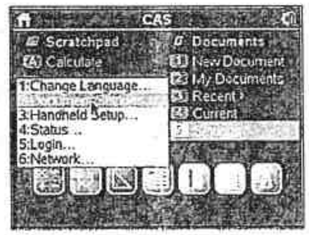
6. $2\cos^2 x + \sin x - 1 = 0$

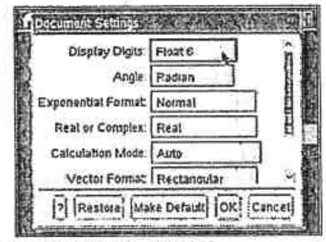
7. Solve for x on $[0,2\pi)$ $\dfrac{x-\pi}{\cos^2 x} < 0$

U. Graphical Solutions to Equations and Inequalities

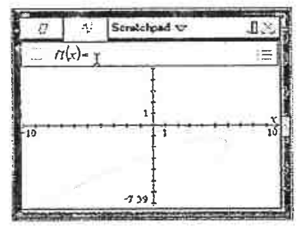
You have a shiny new TI-Nspire graphing calculator So when are we going to use it? So far, no mention has been made of it. Yet, a graphing calculator is a tool that is required on the AP Calculus exam. For about 25% of the exam, a calculator is permitted. So it is vital you are comfortable using it.

There are several settings of the calculator you should make. First, so you don't get into rounding difficulties, it is suggested you set your calculator to show at least three decimal places (and preferably more). This is sandard on the AP Calculs exam, so it's best we start getting used to it. To do this, press the (⌂ on) button, followed by [5] Settings, then 2. Document Settings. Be sure the Display Digits option is set to Float 6. This will ensure that you always see 6 digits across the screen. (There may be times that this can be a problem – i.e. when you have a decimal answer with four or more digits to the left of the decimal. We'll deal with this later) Also, be sure that your calculator's Angle Setting is in Radian mode throughout the year. To make these changes "stick" select Make Default at the bottom.





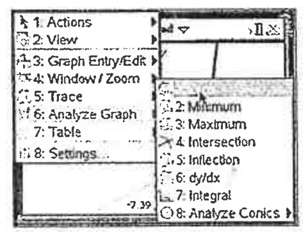
You must know how to graph functions on your TI-Nspire. The best way to graph a function is to press the Calculator key (located in the upper left corner of your handheld just below (esc)) twice on the left side. Notice how each time you press this button, your screen toggles between a calculator page and a graphing page. While on a graphing page, select [tab] to bring up the function entry line. Next input the expression you wish to graph and press Enter



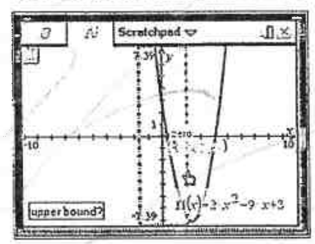
<u>How to find zeros (solutions or *x*-intercepts) of a function.</u>

Step 1: Enter the left side of the equation that's alreay set equal to zero (for example, $2x^2 - 9x + 3 = 0$) into the *f1*(*x*) entry line.

Step 2: Select (menu) followed by 6. Analyze Graph and 1 Zero



Step 3: Move the cursor to a position you feel is left of the zero you wish to find first and press Enter Then move the cursor to the right of the desired zero you wish to find and press Enter You will notice that the ordered pair for the zero will show up automatically once it falls within the range of your lower and upper bound.



<u>How to find the intersection of two functions.</u>

Step 1: Enter each side of the equation (for example, $x^3 = 2x - 3$ into the *f1*(*x*) and *f2*(*x*) entry lines.

Step 2: Select (menu) followed by 6· Analyze Graph and 4· Intersection

Step 3: Repeat Step 3 above.

This problem could also be solved by setting the above equation equal to zero, and using the following procedure "(menu) 6 Analyze Graph and 1 Zero" instead.

Note. You can always move things around on your screen and place them in positions that may make them easier to read. To do this, rub your finger over the Touchpad until the cursor appears. Move the cursor to the item (i.e. intersection ordered pair from the example above) you want to move. An open hand should appear Press Enter to close the hand. By rubbing the Touchpad, the ordered pair should move. Press Enter when you have found a suitable place to put it.