Hello future AP Calculus AB students!

I am looking forward to our class next year. Calculus AB replaces approximately one semester of college calculus, plus some additional topics. The course material is substantially more than any math class you have taken previously, leaving us with very little time to review. Therefore, it is up to you to work through the summer assignment diligently this summer. Your summer assignment is intended to prepare you to begin calculus topics as soon as the school year begins again in the fall. This assignment must be completed and is due the first time we meet, which is currently scheduled for Wednesday 8/30. The assignment is a Fairy Tale with practice problems. Please review the front matter portion first, there is an answer sheet provided for you and you can also show work on a separate sheet of paper as well. It also has links for your reference and a suggested timeline to make the work manageable.

In addition, there is also a packet of pre-calculus problems that you can work through to make sure you are comfortable with the content from previous years. This assignment is optional and will not be collected. I have also included links for those topics as well (under the document tutorial links). I do suggest you review and attempt some of the problems in this packet.

The textbook we use for this course is the 3rd edition of “Calculus: Graphical, Numerical, Algebraic” by Finney, Demana, Waitts, and Kennedy. Here is a link to it on Amazon.
https://www.amazon.com/Calculus-Graphical-Numerical-Algebraic-3rd/dp/0132014084/ref=sr_1_2?crid=3O5S59OUF1V2P&dchild=1&keywords=calculus+finney+demana+waits+kennedy&qid=1622729350&sprefix=finney+demana+%2Caps%2C134&sr=8-2

We have a pdf of the textbook that I will send to you in a separate email. It’s a large file so I don’t want to include it in the same email. This pdf will also be posted on our Veracross page in the fall. You can buy yourself a hard copy of the text if you would like to have that to flip through in addition to the pdf. We will follow the textbook very closely, but I will have copies of all the problems and examples to project in class, so you won’t ever have to bring it to class.

I hope you have a great summer and I’m looking forward to our class together in the fall. Please let me know if you have any questions.

In our first class meeting, we will review course policies, then jump right into our first AP Calculus topic, which will be Limits.

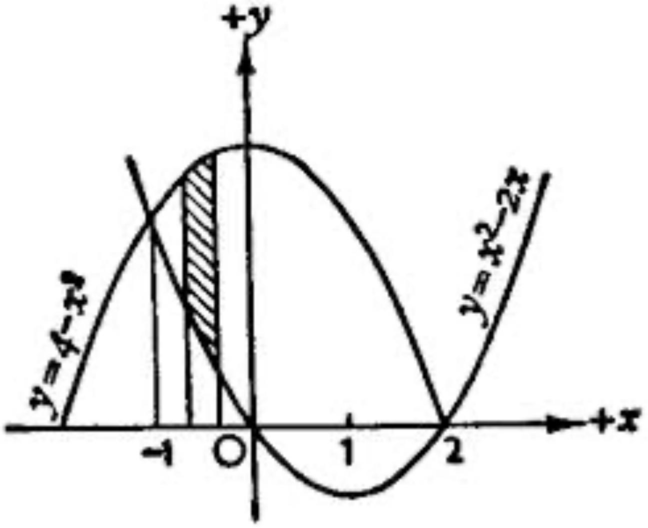
Thank you!
Mrs. Kirkpatrick
kkirkpatrick@johncarroll.org

AP Calculus AB

Summer Assignment

Please complete the attached assignment by ***Wednesday August 30th****. This assignment will count as one of your first assessment grades for this course.*

<u>Following the suggested timeline will spread your workload and make the project manageable.</u>



Do not hesitate to contact me (or each other) with questions on the assignment. I will be available through ***Email****.*

Have a great summer!
See you in September!

Contact Info

Mrs. Kirkpatrick kkirkpatrick@johncarroll.org

Table of Contents

AP Calculus AB Summer Project - Suggested Timeline

Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
June 11	12	13	14 TAKE A WEEK OFF!	15	16	17
June 18	19	20	21 Fairy Tale pages 5-16	22	23	24
25	26	27	28 Fairy Tale Pages 17-27	29	30	**July 1**
2	3	4	5 Fair Tale Pages 28-38	6	7	8
9	10	11	12 Take a Week off	13	14	15
16	17	18	19 Practice Prob. 1-3	20	21	22
23	24	25	26 Practice Prob. 4-6	27	28	29
30	31	**August 1**	2 Practice Prob. 10-12	3	4	5
6	7	8	9 Practice Prob. 13-15	10	11	12
13	14	15	16 Practice Prob. 16-18	17	18	19
20	21	22	23 Practice Prob. 19-21	24	25	26
27	28	29 Last Minute Questions	30 **Project Due**	31	Sept 1	3

MUST BE SUBMITTED BY Wednesday, August 30th at start of class

If you get stuck, try watching these tutorials online.

Tutorial Links

http://www.calculus-help.com/tutorials/

Chapter One: Limits and Continuity
Lesson 1: What Is a Limit?
Lesson 2: When Does a Limit Exist?
Lesson 3: How do you evaluate limits?
Lesson 4: Limits and Infinity

Calculator Use

The graphing calculator is a great tool that allows you to perform various calculations and of course, graph functions. On the AP exam, approximately 38% of the multiple choice questions will allow you the use of a calculator and approximately 33% of the free response questions are calculator active.

Being allowed to use a graphing calculator does not mean that a calculator is necessary to solve the problem. Typically, a fair number of problems that allow the use of a calculator can be solved without one. There are four procedures for which a calculator is expected to be used on an AP exam:

1. *Plot the graph of a function within an arbitrary viewing window.*
2. *Find the zeros of functions (solve equations numerically) – this includes finding intersections of curves.*
3. *Numerically calculate the derivative of a function.*
4. *Numerically calculate the value of a definite integral.*

When asked to justify an answer, the answer cannot be simply "calculator speak" or calculator results – the justification must include mathematical reasoning. Functions, graphs, tables, or other objects that are used in a justification should be clearly identified (i.e. not "the graph" but "the graph of _). Calculus should be used in justifications – explain how you know the answer.

When asked to find calculations that require a calculator you are expected to show not just the answer but the setup (i.e. writing out the definite integral that you are finding). You do not need to write out all steps if it is a free response question where a calculator can be used but you must show what you type into the calculator as well as the answer.

Your calculator should be in ***radian mode*** *and*
answers should be accurate out to ***three decimal places****.*

2023-24 AP Calculus AB Summer Assignment Answer Sheet

Name: ______________________ Date: ______________

If you need more space, please feel free to attach a piece of notebook paper

1.

2.

3.

4.

5.

6.

7.

8.

9.

10.

11.

12.

13.

14.

15.

16.

17.

18.

19.

20.

21.

Summer Assignment MUST BE SUBMITTED BY
Wednesday, August 30th at start of class.

FAIRY TALE CALCULUS:

LIMITS

By Sarah Allen
Illustrated by Kimberly Grau

For my Grandmother.

Contents

The Cook's Apprentice

Once upon a time there was a cook's apprentice who lived in a small kingdom called Cartesia at the base of a great cliff. This cook's apprentice's name was Kip, and he was hopelessly in love with the princess, although he had never spoken to her.

The king and queen of the realm had died many years ago, leaving the king's advisor, Edward, to look after Princess Rhianna and the kingdom until she came of age. Archduke Edward, as he was known, had three strapping sons, who were his pride and joy: Alec, Alex, and Marion.

Soon the time came for Princess Rhianna to marry. The Archduke pleaded with her to consider one of his sons, but, truthfully, although each was more handsome than the next, she did not love any of them. So, she sent word to all the nearest realms, requesting suitors. Soon they came in droves.

Some were too silly, some were too scholarly, and some were too warlike. After many weeks, when the princess had nearly given up hope and was just starting to consider how she might choose between Alec, Alex, and Marion, Prince Henry from the Western Isles arrived. He had come the farthest of any, and he was everything a King should be: wise, kind, and strong.

Prince Henry and Princess Rhianna

discussed possible arrangements late into that evening, and all the next day, and on the evening of that second day they announced their betrothal.

The archduke was furious when he heard of the engagement. How could the princess choose some foreigner she had known for only a few days over one of his sons, whom she had known since she was a child? His sons were handsome, and strong, and he had seen to their educations in every way possible. They knew every ounce of etiquette there was to know, every sword thrust and parry, every battle cry, army formation, and siege tactic. But more than that they knew this kingdom, and they knew what was best for it. He knew what was best for it. That very evening, he went to see the Sorcerer of the Wall.

The Sorcerer of the Wall was a strange man, a hermit, who lived far outside the city. Although it was late, and dangerous outside the walls, the archduke went alone. It was evening, and the sun was setting behind him. The shadow of the castle stretched out far in front of him. He walked along a small dirt path; to his left the sheer face of the cliff rose hundreds of feet straight up, and to his right the great plains stretched out as far as he could see, glowing orange in the fading light.

At last he came to a narrow ledge that climbed up the side of the cliff. Looking all around to see if he was observed, he started up the ledge. The path rose higher and higher, until he could no longer bring himself to look down. Once he glanced up, and saw a flicker of unearthly green light. After that he glanced up no more.

Finally he came to an outcropping of rock.

Here there was a small wooden door set into the rock. He knocked three times and stepped back.

Edward heard soft echoing footsteps approach, and the door opened. A man with very old eyes and a long beard opened the door. His back was straight, and he was dressed in simple brown robes.

Edward explained that the kingdom was in danger. The princess was going to marry some unknown foreigner who would destroy their little kingdom, maybe enslave everyone and turn the land into a mining outpost. The Cartesian Mines were already famous throughout the lands. How much more famous would they be if they were mined by slaves who were forced to work day in and day out, rather than well-paid laborers who went home to their families every night?

The sorcerer listened to Edward's diatribe with heavy eyes and an expressionless face. At last, when Edward had finished, the sorcerer reached into his pocket, and drew forth a piece of parchment and a stick of charcoal. With the stick he drew three horizontal lines on the parchment.

"Write the name of the person who causes you trouble on one of these lines, and your wish will be granted. Just beware, no magic comes without a cost." He handed the paper and the charcoal to the Archduke and shut the door softly.

Buoyed up by hope, the Archduke slipped the items into his pocket, and hurried home. That evening, he wrote a name on the first line of the parchment.

The next morning, an alarm was raised in the castle. The princess had fallen deathly ill, and Alec, the oldest of Archduke Edward's sons, was found dead.

It was too much to be a coincidence. Everyone suspected they had been poisoned. Prince Henry called a meeting of his own people, and asked the castle guards to attend as well. He had already introduced himself to everyone in the castle, and was well-liked and respected, so when he stood to address the crowd in the great hall, everyone listened.

"It appears we have a traitor in our midst, someone conspires to murder the princess-"

The archduke stood, and all heads turned towards him as he spoke. He was dressed all in black, and his face was gaunt and gray. "How do we know it wasn't you? My son was the most likely candidate for the throne. Maybe you wanted him out of the way."

There were whispers among the crowd, but no one interrupted Prince Henry when he spoke. "True, except that the princess and I were already engaged to be married. I would gain nothing by poisoning her."

The archduke opened his mouth in an angry snarl, but before he could speak the prince cut him off with a quick swipe of his hand. "I know you are filled with grief for your son, archduke. I promise to you and to everyone that I will find out who is responsible for poisoning Princess Rhianna and your son. I will not rest until I find a way to cure the princess, even if I cannot bring back Sir Alec. I have heard tale of a powerful sorcerer who lives near here. I will find him and ask him for his help. We will leave first thing tomorrow morning."

The next morning, however, when the page boy went to find Prince Henry for the expedition, he found him still asleep, his face grey and his lips blue, and he could not awaken him. Sir Alex, the second son of the archduke, was also found dead.

This time, the court champion, whose job it was to go on quests for the realm, and who had been slightly annoyed at the visiting prince's presumption yesterday, declared that he would embark on a quest, and save the prince and princess from this sickness. He left right away, before anyone else could jump in and try to steal his glory, and so no one knew when the next morning, camped at the base of the wall near the sorcerer's home, he did not wake up. That morning, the youngest of Archduke Edward's sons, Marion, was found dead.

The kingdom turned to the grief-stricken archduke, who, with a heavy heart, declared that he would take charge of the kingdom of Cartesia while they waited for the champion to return with a cure for the sleeping prince and princess.

When Kip heard the news of the princess' sickness, he was so filled with anger and fear that he couldn't concentrate on his cooking duties at all. To be fair, he didn't usually concentrate very hard on his cooking duties, but this was especially bad. The head cook yelled at him several times, and even threw a pot at him, which was new, but all Kip could think about was who might have poisoned the princess. When he asked the other cook's assistants, they just shrugged. When he asked the guards, they sneered at him condescendingly, or ignored him, or, if he was particularly persistent, threatened him.

When the prince took ill as well, Kip's fear increased, as did his feeling of needing to do something. He stopped going to the kitchen at all. The cook had escalated from mostly yelling to mostly throwing things, so he figured he shouldn't go back anyway.

He stood on the battlements of the castle with everyone else, waving banners and handkerchiefs as they watched the champion ride out on his quest, but he didn't feel much hope. And, the next morning, when the Archduke's third son was found dead, and no one else was found in a wakeless slumber, he knew he had to do something.

And so he snuck back to the kitchens late that night when no one else was there, and packed a small backpack with a loaf of bread, an apple, and a piece of cheese. Then, quietly, he set out in the direction the champion had taken.

Just as the sun was rising, he smelled the smoke of a campfire, and not long after that he came to the camp of the champion. The fire had burned down to ashes. Next to it lay the champion, his face grey and his lips blue, his hands clasped around the hilt of his sword, as if even in sleep he had felt some magical enchantment come upon himself and had attempted to draw his sword before succumbing to it.

Kip looked at the sword, but it was much too large for him to carry. He rifled through the champion's travel sack and found some dried meat and some fancy cider, both of which he took. He also found an extra blanket, which he tucked around the sleeping champion. Then he looked around.

Up ahead he saw a narrow ledge that climbed up the wall. He had never been up the wall, had never even been near it. Like everyone else, he had heard the stories. There were things that lived on the wall. Those who went up never came back, or if they did they were not the same. But something within him told him that that was the way the champion had been planning to go, and so it was the only way for him to go. So, thinking of the princess, he lifted his pack onto his shoulders and started up the ledge.

The trail climbed quickly, and after a short time Kip was higher in the air than he had ever been before. Before he could think about this too much, luckily, it widened into a ledge, and there, sitting on the edge of the ledge was a man. He looked young, but he had a long gray beard which dangled over the edge of the cliff between his feet. Kip approached him cautiously.

“Er, excuse me, sir?”

The man heaved a great sigh. Without even turning he said “Yes, what is it? Your stepmother? Your older brother? Maybe he inherited everything and you will be left with nothing if you don’t get him out of the way somehow?” Gloomily, he kicked at the end of his beard.

“No, I’m just a cook’s apprentice. My name is Kip, and the princess of Cartesia has fallen under some spell. She is asleep and can’t be woken, and I’m looking for some way to wake her up. Can you help me?”

At this the man turned and looked Kip in the eye. Then he smiled.

“Ah, yes! That I can help you with. Although, I’m afraid it will be very difficult. Come inside please.” Then he disappeared.

Kip looked around, and for the first time noticed a door set into the cliff side. He pulled it open and saw a rough stone tunnel. A voice echoed up from somewhere deep inside. “Hurry up! We don’t have all day!”

Kip closed the door behind him, and immediately light sprung up around him. He couldn’t tell where it was coming from, but it followed him along as he walked down the tunnel. He passed several smaller side tunnels, but when he tried to peek into these the light didn’t follow him, so he continued along the main path until it opened into a large, cozy room filled with strange knick-knacks. In the center was a large wooden table with three chairs, and on the floor were several rugs. A fire burned in a stone fireplace at one end. Kip took a few steps into the room and then stopped.

“Ah, good!” Came a voice behind him. Kip jumped and turned to see the man with an armload of what looked like silver and jewels. These he dumped on the table; then he pulled two mugs of something steaming from somewhere and motioned Kip to sit.

“Do you know what happened to the princess?” Kip asked.

“Yes, of course. I enchanted her.” The sorcerer said.

“You? But, why would you help me un-enchant her then?”

The sorcerer sat down, and heaved a heavy sigh. Then he took a long drink from his mug, and sat staring into it disconsolately.

“You don’t have much time, but since you ask, I’ll tell you my story. It’s not a long one, and no one ever asks. Many, many years ago I was a young prince. I am actually Princess Rhianna’s great, great, great, grand uncle. In those days, there was more magic in Cartesia. There were wizards and sorceresses. We had a sorcerer at court. His name was Arman. From the time I was a small child I wanted nothing more than to be a sorcerer myself. I begged Arman to teach me, and to his credit he tried, but I just didn’t have the gift for it. I couldn’t control the elements, or talk to animals, or brew potions. I excelled at most things. I was a brilliant swordsman and tactical strategist and I always knew the right thing to say to smooth over hurt feelings, which just made it all the more terrible that magic seemed to be completely out of my grasp.

“When I turned sixteen, I left home. I vowed that I would not return until I had learned how to be a sorcerer. At first I went south, across the plains. I met many people, magical and non-magical, there, but no one who could help me. Next I went East, and there I met no people but encountered many beasts and animals, but none of them could give me magic. So, I went West, and here there was no one at all, only wasteland. At last I resolved to go up the wall. Really, I had known all along that that was what I must do, but even in those days of magic, no one went up the wall. It is not a place for men. But, so determined was I to become magical that I tried it anyway. I found a path and started up it. No sooner had I begun than a beautiful fairy appeared before me. She was dressed all in green, but she herself seemed to be made of fire, and she warned me to go no further. But I could not be convinced. As soon as she was gone I continued on."

He paused for so long that Kip wondered if he had fallen asleep, or died. But, at last he said, "And that's my story. I won't tell you what I found there. You can see that in at least some ways I found what I sought. I am magical. But, it is not something I can control. The magic that I have lives inside me, and it does whatever anyone asks of it. And most of what people ask of it, I've found, is terrible. So I live alone, as far up the wall as I dare. The magic likes being here, and most people are too scared to come this way."

"That sounds terrible," said Kip.

"Anyway," the sorcerer said, "I will happily help you undo that latest enchantment. It was a nasty one."

"Will the princess live?"

"She and the others who were enchanted will die within ten days unless they are given the cure to the magical poisoning the received."

"What is the cure?"

"It's called a moon flower," the sorcerer said. He pulled out a small, dusty, leather-bound book and opened to one of the last pages. There Kip saw a small, delicate, white flower.

"Where can I find it?"

"At the top of the wall."

Kip stared at the wizard, open-mouthed. "But..." he trailed off.

"Yes, it only grows at the top of the wall. So I've read. I've never been there myself. I didn't make it very far up."

"I…didn't you just tell me that no one should go up the wall?"

"I have read about the wall for many years, and there may be a way. By the way, its proper name is the Cartesian Plain."

"Plain? How can a wall be a plain? It's vertical. And nothing grows on it."

"I don't know, maybe it means plain like not decorative. Anyway, there is a path that you might take. It is called the Limited Path."

"The Limited Path?"

"Yes. There are many, many paths along the wall, going up and down and left and right, all travelled by invisible beings and creatures. You must go the way the creatures go, if you can. There are many paths that cannot be walked by men. You can walk only in places where there are limits."

"What does that mean? And, how do I follow the creatures if they're invisible?"

"There are holes in the paths, and places where they jump suddenly from one place to another. The creatures skip over these holes, and you can skip over some of them, too. Unless the path jumps. If it jumps you'll have to go back and find another path."

From his sleeve, the sorcerer pulled a crystal, it was thin and wide and had a faint rosy tinge to it. "This will let you see the creatures. As for what it means, I'm only telling you what I've read. I can't say I understand it either. A word of warning, though, if you make it to the top of the wall, you must not look through the crystal there." He handed it to Kip, who put it into his pocket.

"Now," the sorcerer turned to the pile of jewels on the table, "The magical beings have many enchantments, so that even with the crystal you might not be able to see them. To help you on your journey, I will give you these three objects as well."

He picked one up and handed it to Kip. It looked like two silver spheres that had stuck together and were being slowly pulled apart. The silver was a little tarnished with age, and it was set with tiny rubies. Kip put it carefully into his backpack. Next, the sorcerer handed him a silver square, set with tourmalines. Lastly came a set of silver scales. In the middle of the scale was a tiny arrow. As the scales moved, the arrow rotated so that it pointed at different little numbers.

The sorcerer brushed his hands off, drank the last of the liquid in his mug (Kip still hadn't touched his,) and stood up. "All right, that's it, you'd better be on your way!"

Kip lifted his pack to his shoulder. It was much heavier and bulkier now, and something hard poked him in the back. The sorcerer patted him on the shoulder. "Good luck now, and don't look so serious. You seem like a likely lad, I'm sure you can do it." His eyes darted to the side as he said this. His pats turned into nudges which propelled Kip back up through the tunnel and out the door. Kip was just turning around to say something when it slammed shut.

He stared at the door for a moment, and had just about decided to be on his way when it opened again.

"I forgot to tell you where to start!" The sorcerer bustled out past Kip. "Come, come, this way. You start at the origin." He led the way back down the path to ground, and then further along the wall for about ten minutes. There they reached a large, wide set of stairs set into the wall.

"This is the origin. Not all paths start here, but many do, or at least they pass through this point. It's a good place to start. Be careful. Some paths go on forever. Some go up and up forever but never reach the top. Some go down and down forever but never reach the bottom. They get close enough that you could jump down, though, but I wouldn't try it. It's never good to leave the path."

The sorcerer stood aside, and seemed to be waiting for Kip to start up the stairs. So, uncertainly, Kip took a step onto the path. He didn't feel anything. So, he took another step.

"Good luck! Remember, don't use the crystal at the top! If you get there! Which I'm sure you will!"

Kip turned but the sorcerer was already gone.

For a moment, Kip considered turning back. He could go back to Cartesia, maybe sell the magical items the sorcerer had given him. If they were really silver and rubies and tourmalines, he could probably make enough money to live comfortably for a long time, maybe buy himself a little house with a garden. He could grow vegetables and cook just for himself. But, then he thought of the princess. She would die in just a few days- seven, if he was counting correctly- unless he found the moon flower for her. With a sigh he turned back to the steps and pulled the crystal from his pocket.

He held the crystal up in front of one eye and looked through it at the bare rock. The rocks were distorted by the unevenness of the crystal, and at first he couldn't see much. Then, he noticed small flashes of golden light, and smaller little eddies of green light. As he watched, a large black shadow moved into view, and past him, up the stairs. With a shiver he put the crystal back in his pocket. The stairs were ordinary, rough-hewn rock steps now, and empty. He readjusted his pack straps and started up.

Kip was not used to climbing stairs, and soon his legs were burning. His pace slowed and slowed and he used his arms to push on his knees to help him up. He was concentrating so hard on each step that he almost didn't notice when he came to a fork in the road. The wide steps split into two, one continuing nearly straight up, the other branching off just slightly to the right of that. He took off his pack and sat on the ground for a moment to rest as he looked at the two paths. He pulled the apple out of his pack and ate it as he considered.

Reluctantly, he pulled out the crystal again and looked. This time the flashes of light were a little more distinct. He could almost see shapes, strange people and animals, hurrying along up and down both sets of stairs. They ignored him. He found he couldn't look at one for long without having to look away.

Not knowing if either of these was the limited path, he decided to continue up the steepest path, hoping it would lead him to the top the quickest.

Before he could take a single step, there was a sound like rain falling outside a window, and a beautiful fairy appeared before him. She seemed to be made of water, all except her eyes, which were green.

"Who are you and what are you doing here?" she asked.

"My name is Kip and I am looking for the moon flower to wake the princess, Rhianna."

"You are not welcome here," she said. "There are no moon flowers here. Go back to where you came from."

"I'm sorry," said Kip. "I don't want to be here, but I was told that the only way to save her was to climb to the top of the wall and find a moon flower there. Do you know which stair will take me to the top?"

"They both go there." She smiled at him, showing rows of sharp teeth. Then she put her palms flat together and drew her hands apart. Between them was a large bubble of water. Quickly, she thrust her hands back together again. The bubble popped, and she was gone.

Something was different. The air was darker, a faint, cold, wind was blowing. Kip looked up and saw that dark clouds had moved in, and the rocks were just showing the first speckles of rain. He had better hurry. He looked for the steeper path, but it was gone. Looking all around, he realized he didn't know where he was. He was still on the wall, but someplace completely different. The stairs were gone, and instead there was only a smooth, narrow ledge that stretched along horizontally.

He looked left and right, finally decided that the path climbed slightly to the right, and so he set off. He wondered if he was still on the limited path. If he could walk here, he supposed he must be.

He walked for several hours. The sky grew darker and the rain grew heavier. He wished he had brought something warmer to wear, or matches for making a fire, although there wasn't anything to burn anyway.

It had grown so dark that he almost didn't see that the path ended before he put his foot down on nothingness. Heart pounding, he backed up and looked at it more carefully. The path looked like it ended in a sheer drop-off, but it was so dark that when he stood at the very edge and squinted he couldn't tell if there was anything on the other side. Looking up, he saw something strange. It looked like the path, but it was about ten feet above him. He couldn't tell if it continued very far up there, but he could see the beginnings of a ledge.

Kip wondered whether he should try to climb up to that part of the path, but then he remembered what the sorcerer had said about holes. Sometimes there are holes that he could skip over; sometimes there are jumps. He pulled out the crystal and looked through it.

The creatures here were all the same, they were dark, heavy blobs. He could see lights glowing from deep inside each of them, though. These lights were different colors, and they were arranged in two rows, one above the other. He watched as a beast approached the place where the path ended, but just as it got there, it disappeared.

There were beasts coming from both directions along the path, so he knew that it continued somehow, but he didn't know if it continued in a way that he could follow.

After watching for a while, feeling frustrated as each beast disappeared just as reached the place where he needed see where they went, he noticed something. The heart lights all had something in common. There was always one light repeated in the top row and the bottom row, and just before they disappeared, these pairs twinkled slightly.

This reminded him of the two spheres stuck together, so reached into his pack, which was now soaked, and pulled it out. A beast was just walking past him, and he held the object out towards it as far as he dared. This beast had a red light on top and on bottom. Kip stretched out just a little farther, and the sphere grazed the side of the creature. The red lights went out. The shadow shook itself, but continued on just as it had before. Kip followed behind it eagerly as it approached the cliff's edge.

When it got there, it simply walked across the hole, as if it were stepping across a tiny crack, as if it weren't even there, and now Kip could see that the path continued on the other side. He held tightly to the silver spheres, which glowed redly now, and stepped over the hole. His foot connected with firm ground on the other side, and he stepped across.

Looking back, the hole looked exactly the same from this side as it had from the other. Sliding the spheres back into his pack, Kip took a deep breath and continued on.

Finally Kip came to another fork in the path. Here, a set of steep stairs intersected the flat ledge he had been on. Looking through the crystal, he saw many fairies, red and blue and green and yellow, flitting up and down the stairs, dodging among the shadow beasts of this path. So, he turned up the stairs and began to climb.

The path climbed higher and higher. Finally the rain stopped and the clouds parted and the sun came out again. In a way, this was worse, because now Kip could see how high he was. Looking behind him, there was a steep drop down hundreds of feet. Off to the left he could just see the edges of Cartesia, bathed in afternoon light. The castle was out of his view, hidden by an outcropping of rock, but he could see tiny little walls and stone houses. He squinted, but couldn't make out any people.

Kip continued on and soon he came to another fork in the path. Again he took the steeper of the two paths. Not long after that he came to another hole. He set his pack down, and pulled out the cider. As he sipped it, he looked through the crystal. The creatures here looked like some kind of cross between a spider and a grasshopper. As the came to the hole, they shot upwards, and landed on an outcropping of rock. As they shot upwards, they changed shapes, some grew legs or arms, others grew bigger or smaller. Other creatures were coming the opposite direction, too, and as they jumped down to Kip's part of the path they changed, too.

Clearly, this was not one of those holes you could just jump across. Reluctantly, Kip turned back the way he had come, and when he came back to the fork in the path, he took the path that wasn't quite as steep.

He climbed until the sun began to set, its last rays glancing along the edge of the cliff, sending long shadows. At last, when it became too hard to see, he found a small sheltered spot in the wall, and climbed in. He ate a little of the champion's dried meat, and quickly fell asleep.

He awoke stiff and cold the next morning, and he started on his way immediately, without eating anything for breakfast. The clouds were lower today than they had been yesterday, and soon he climbed up into a thick fog. He had to tread carefully, testing out each step. Soon he came to another hole in the path.

Watching through the crystal, he again saw the shapeless black monsters of the day before, only these had rows of square lights in their hearts. None of the lights on the two rows matched this time, though.

At a loss, Kip pulled out the silver square set with tourmalines. Gingerly, he held it out to the next passing beast. As the square brushed its side, the lights changed, but when it came to the hole it still disappeared. Kip tried again on the next creature. Again, when the square touched it the lights changed, but still it disappeared. On the third try, Kip noticed that when the lights changed there were matching ones on the top and bottom again, so he pulled out the set of spheres.

On the next beast, Kip first touched it with the square. The lights changed. Now there was a soft blue light on top and bottom. Kip touched it with the spheres, just before it reached the hole. This time, he saw it walk straight across.

Grinning, Kip picked his pack back up, placed the square and the spheres back into it, and crossed the hole and continued on.

He climbed for several hours more. Slowly, the fog thinned and the sun came out again. Kip continued to climb. Glancing down for the first time in several hours, he saw that he was above the clouds now. The sun was bright and hot, and just a few hundred feet below him was a soft, magical world. It looked solid enough to stand on.

Looking up, he craned his neck, trying to see if he could see the top. There, just at the limit of what he could see, there looked like an edge.

Kip continued to climb, and the wind picked up and started to blow in earnest. His skin felt raw, burned by the sun and dried by the wind. His lips started to crack. He climbed and climbed but the edge seemed no closer.

Finally, he pulled out the scale. He held it out in front of him, pointing towards the path. The two plates moved up and down, the arrow twisted, finally it settled down. The arrow pointed to a number: 9,045.

Not knowing what to make of this, Kip put the scale away and continued on. The farther he walked, the flatter the path got and the less vertical progress he was making. But, there were no other paths, and he had gone a long way since the last intersection, so he didn't want to turn around.

When the sun was directly overhead, he sat and rested for a while. He ate the last of the dried meat and half the loaf of bread. He pulled out the scales again. Again they wobbled for a little while, and then settled down with the arrow pointing at 9,045.

He continued walking. Now the path was flat, as far as Kip could tell. He wondered how high he was. Even the clouds looked a long way down now.

At last Kip came to a wide intersection. Here, six different paths crossed each other. Kip stood in the middle, trying to decide which way to go. He held the scales out to each of them in turn. There was a path that sloped gradually downward, when he held the scales out to it they read 317. Another path shot steeply downward, this one read 0. Another path that angled up also read 0, though, so he wondered what that meant. Finally, he pointed the scales at a path that looked fairly flat, and the arrow pointed straight up. Here there was no number, it just pointed up. Kip packed the scale away and took that path.

Even though it had started out flat, this path quickly angled upwards. The farther Kip walked, the steeper it got. The wind blew harder and harder, dust from the rocks got into his eyes. His throat was dry. He hadn't had anything to drink since the cider the day before.

But the edge looked closer now. He was definitely getting closer.

Finally, as the sun was just beginning to set, he came around a corner and saw a short set of silver stairs. They looked new and perfect and they shone brightly. Kip looked through his crystal, and he saw the water fairy sitting there watching him. Her eyes were black and angry.

"These are my stairs," she said. The wind was so loud that Kip couldn't hear them, but they jangled in his mind anyway.

She stood and took a step toward him.

"You have climbed the stairs. You have

walked only the limited path. You may reach the top, but first, you must give me something of equal value to that which you seek."

Kip thought about the things in his pack. One by one he offered them to her. But at each she shook her head.

"None of those are powerful enough. You must give me that which allowed you to get this far. Something powerful enough to make you climb the wall."

For the first time, Kip looked through the crystal at himself, and he saw in his heart many small colored lights. One, though, shone large and golden. His love for the princess. That was the most powerful thing he had.

"But, if I give you that..."

She shrugged. "That is my offer. Give me that and I will let you pass."

At last, Kip nodded. The fairy reached out a watery hand, he felt it pass through his chest, felt its fingers wrap around a warm glow in his heart, and slowly draw it out of him, leaving a hollow space behind. Then, she was gone. And before him were the stairs.

Kip climbed the few stairs easily, and found himself in a wide, flat meadow. It was filled with flowers. None were flowers he recognized, all were more deeply colored and more intricate than any he had ever seen before. But none were the delicate white moon flower he was looking for. The light had faded to the soft purple of evening, and he walked slowly to the edge of the cliff. There he sat with his feet dangling over the edge.

The clouds had cleared again, and far, far, far below him was the ground. Lifting his head, for the first time he looked off into the distance. The grassy plains stretched for miles and miles, but past those he could see purple mountains, and forests, and rivers. To the west he saw the wasteland the sorcerer had mentioned, but past that he thought he saw silver spires- some great city, and past that a thin line of blue water. Something warm filled his chest, something larger and truer than what had been there before.

He sat for a long while. He sat as the stars came out above him, many more than he had ever seen before. At last the moon rose, white and full and so much closer seeming. When it had lifted up fully from the horizon, Kip turned away from the view and stood. There in the meadow were a million soft white flowers, glowing in the moonlight. Kip picked three and put them in his pack. Then he fell into a deep and dreamless sleep.

When he awoke it was already late morning. He stretched, ate the last of the loaf of bread and the piece of cheese, and set off down the cliff. He came to many forks, and each time picked the steepest. He came often to holes and jumps and places where he could cross and places where he had to backtrack. When he came to large intersections, he checked the scales. It seemed that the scales told him what the path would do eventually. Some of the paths, like the sorcerer had said, levelled off, and the scales told him where they levelled off. He avoided any of these, even the ones that said they levelled off at zero. Instead, he took only the paths for which the arrow pointed straight down.

The journey to the bottom of the wall seemed to take no time at all. Before long he was back in the fog, and then he came out underneath it, where it was raining again. He went as quickly as he could without slipping on the wet rocks. He passed a small waterfall at one point, but was too afraid to drink the water, even though he was very thirsty.

At long last he found himself at the base of the wall. It wasn't the same place he had started out at, and he wasn't sure exactly where he was, but he set off in the direction he thought was west. He passed the sorcerer's home and soon came to the sleeping champion. His face was grayer and his lips bluer than before.

Kip pulled the first flower out of his backpack and set it on the man's chest. Immediately, a soft white light poured out of it, and it wilted. Color returned to the man's face, and he began to breathe.

The champion took a deep, gasping breath and sat up, opening his eyes. He shook his head and rubbed his temples for a moment, and then he looked up and saw Kip. Then he saw the wilted flower on his chest. He looked in wonderment at the boy.

Kip took the two remaining flowers from his pack, and held them out to the champion.

"These will cure the prince and princess. Just place them on their chests."

The champion just stared at him, open mouthed, for a few moments, but he didn't take the flowers. "How long have I been asleep?" he asked finally.

"A few days. You'd better hurry. The princess only has a few days left. If she doesn't get this flower, she'll die."

"Then you'd better get it to her," the champion said. At this, he stood up, and, leaving everything but his sword, he helped Kip to stand. "Put those back in your pack. We wouldn't want anything to happen to them." He nodded towards the flowers.

Kip put the flowers carefully back in his pack. Then, he tried to stand, but found that he couldn't. His legs had given out at last, and he was too tired and sore to walk. The champion seemed to see this, and he reached down and gently helped Kip up. Then, he lifted him up into his arms, and began to walk.

It took them a long time to reach the castle. Just outside the castle gates the champion stopped and put Kip down.

"There is someone powerful here who put these spells on the prince and princess and me. When the people see me returning they will think I have the cure. We had better split up. Can you walk, lad?"

Kip wasn't sure if he could or not, but he nodded anyway and, when the champion put him down, he found that he could shuffle along a little, although every part of his body hurt.

The champion gave a quick nod. "I'll distract them. You find the prince and princess. Good luck," he said, and strode off toward the castle.

Kip waited a few minutes. Soon he heard shouting, and running footsteps. After a few minutes, he walked unobtrusively in through the gates. He went quickly to the side entrance, where the kitchens were, and slipped inside.

Unfortunately, he ran almost immediately into the head cook. Their eyes met and the cook's face darkened in fury.

"Stealing!" was all he managed to get out. He grabbed Kip's arm, who was too weak to resist, and started dragging him off down the hall.

"Wait, sir, please. I have something; I have the cure for the princess," Kip said.

The cook's mouth curled in contempt. "And lies. I can't take any of this anymore. Fired. Dungeons. Lashings." Now, unable to make full sentences, he was just saying threatening words, and still dragging Kip down the passageway.

"Please, I can prove it. I can give you things," Kip said.

The cook stopped. His eyes narrowed at Kip.

"Here, in my backpack." Kip squirmed around until he could reach inside, and he pulled out the first thing his fingers closed around, which was the silver spheres. He handed it to the cook.

"Here, you can have this. A sorcerer gave it to me."

The cook raised his eyebrows skeptically. "Just because you stole things from other people, too, doesn't mean I'm going to believe you now." But his grip on Kip's arm loosened a little.

"And you can have this too, and this." He pulled out the square and the silver scales, too, and handed them both to the cook. "Just please let me go. I promise I'll come back later so you can punish me."

The cook had to let go of Kip in order to hold all the silver and jewels. He had opened his mouth to speak, but couldn't seem to get any words out.

Kip took advantage of this, and, feeling his muscles scream in protest, he ducked around the cook and sprinted as fast as he could down the corridor.

“Hey!” the cook shouted halfheartedly at him, but didn’t follow.

Kip stopped running; he tried to walk quickly, but none of his muscles seemed to be working properly. He limped as quickly as he could, twisting and turning through back hallways, until he came to the servant’s entrance to the princess’ quarters. He put his ear to the door, and listened. When he didn’t hear anything for several minutes he pushed the door open softly. There, across the room, on an enormous bed, lay the sleeping princess. Her face was ashen, and there were dark circles under her eyes. Kip couldn’t see if she was breathing. He started to cross the floor toward her, but stopped when he saw there was someone else in the room. Archduke Edward.

His eyes were gaunt and hollow, and his face was streaked with tears.

He and Kip looked at each other for a moment. Then Kip turned back to the princess. He pulled the second flower from his pack, and laid it above her heart. It wilted, and silvery light flowed into her. Heat came back into her face, and she sat up, coughing, and opened her eyes.

In the corner, Edward gasped.

Princess Rhianna looked around, confused. “What’s going on?” she asked.

Kip heard Edward coming towards her. He spun around, although he would have no idea how to defend her, but Edward passed him by and knelt by the bed. He bowed his head and cried.

"Forgive me, Princess. I tried to poison you. I thought I wanted one of my sons to be king, and I poisoned you thinking that if you were gone one of them would be next in line for the throne. But when my first son died I realized that I didn't want the throne for them, I wanted it for myself. And when the Prince said he was going to go look for a cure for you I knew he would find it, and so I poisoned him too. Even though-" He choked on a sob and could not continue.

The Princess looked from Edward to Kip. Her color was returning, and she looked more beautiful than Kip had ever seen her, but he felt nothing.

"Who are you?" she asked.

"Kip, your majesty. I work in the kitchens."

She looked down at the flower on her chest. "And you brought this and saved me?"

He nodded. That reminded him.

"And, actually, if it's all right with your majesty, I have another flower here for Prince Henry."

She nodded, pulled off the covers, and stood up. Edward was still crying, kneeling on the floor.

"Please wait here, archduke," she said, then nodded to Kip. "This way, Kip." She strode across the room, pulled open the door, and swept through it. Kip followed as best he could, but he had trouble keeping up with her as she swept through the halls. Then she jerked to a stop, and pulled open another door. Inside was another bed, and on it lay Prince Henry.

Kip walked forward, pulled out the last flower, and lay it upon his chest. Then he sat on the floor, laid down, and closed his eyes. He heard the princess saying something, but couldn't make out what it was before he fell asleep.

When Kip awoke it was several days later. The cook was sitting by his bedside. When he saw Kip was awake he first offered him some water, then offered him some food. Then he shook his head and said, "I can't believe I almost stopped you. And I can't believe I was mad at you for stealing food."

Kip smiled, "Well, I probably wouldn't have believed me either."

The cook shrugged. He asked Kip where he got the flowers from, but Kip realized he didn't want to talk about the wall, or the magic, or any of his journey.

Three days later, Princess Rhianna and Prince Henry were married in a small ceremony in the castle. Two days after that, they came to talk to Kip, who was still recovering in bed, visited frequently by the head cook.

They offered him lands and a title, gold, jewels, whatever he wanted. Kip said he was grateful, but that all he wanted was to go and explore the rest of the world- the places he had seen from the top of the wall, although he didn't say that.

So, they gave him as much money and food and supplies as he could carry, and a strong horse to ride, and said that he would always have a place there at the castle.

Before he left, Kip asked them for two more favors.

The first was that they invite the sorcerer to live in the castle. They hired a secretary for him to interview people who would ask for magical help, so that the secretary could decide whose requests were worthy. It wasn't foolproof, sometimes people lied well enough to convince the secretary and then were able to ask the sorcerer for evil things, but, for the most part, from then on he was able to do more good than harm in the world, which was all he really wanted, and all he felt he could ask for.

The second was that they hired ten more kitchen boys, and two assistant head chefs. From then on the cook had all the help he needed, and he never yelled or threw pots at anyone ever again.

As for Kip, he rode off into other lands and met many people and had many adventures. He thought of the Princess from time to time, and always with a small feeling of emptiness, as if he couldn't quite grasp something that should have been there. But, in time he found other loves and his heart was filled again. And everyone lived happily ever after.

THE END

The Math

Limits are one of the foundational ideas in calculus. They're really interesting, but they can be kind of strange, too. In this story, the magical creatures on the wall represented mathematical functions (like $y = x^2$ or $y = 3x + 7$,) and in order to follow the correct path, Kip had to figure out what they were doing at certain points. When we try to figure out what a function does at a certain point, we mean that we want to find what the 'y' value does when the 'x' value gets closer and closer to some number. The interesting thing is that it doesn't matter what the value of the function actually is at that x value. All that matters is what it looks like it's going to do as you get closer and closer to it. This is called ‘finding the limit’ or ‘taking the limit.’

There are lots of different tricks for finding limits. Some limits are easy. You just plug the x value into the function and solve for y and that’s the limit. Some are harder, though; like when the magical creatures disappeared right before they got to a certain point.

Some of these are called ‘holes,’ which is when a function doesn’t have a y value at that point. Even if the function doesn’t have a y value we can still figure out what the y value would be if it had a point there. Sometimes we do this by factoring and cancelling (represented by the double sphere,) and sometimes we do this by multiplying by square roots (represented by the square.)

We can also figure out what functions do when their x values get really big. This is called taking the limit as x goes to infinity. This was represented by the scales in the story. The scales showed Kip whether the function would eventually level off or whether it would go all the way up or down.

Sometimes functions don’t have limits. If they jump from one value to another, then there is no limit at that point. This is called a ‘jump discontinuity.’ Kip was told by the sorcerer that he had to follow the ‘limited path’ which meant that he could only walk in places where there were limits. Since there is no limit at a jump discontinuity, Kip had to turn around.

The next couple of chapters go into all the specifics of the algebra and the notation. So, if you’d like to see how the story relates to the math and how to actually calculate limits, read on!

Limits

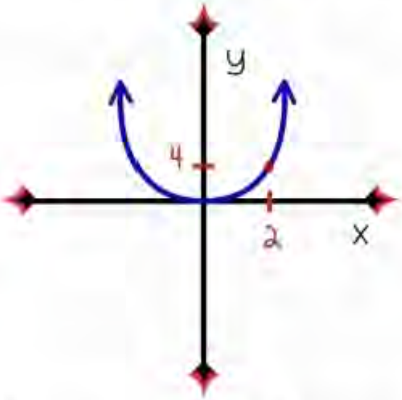
Limits are one of the first things you cover in calculus. Basically, when we find the limit of a function, we predict what the y value will become when the x value gets closer and closer to some number.

Here's how we write limits:

$$\lim_{x \to 2} x^2 = 4$$

This is read as “the limit of x squared as x approaches 2 equals 4." What this means is that, for the x^2 function, as x gets closer and closer to 2, y gets closer and closer to 4. Which makes sense, because 2^2 is 4.

Here is what this looks like on a graph:



The red dot on the curve is the point (2, 4). You can see that as the x value gets closer and closer to 2, the y value will get closer and closer to 4.

So, the first and most basic kind of limit is pretty easy. You just plug the x value into the function and see what the y value is.

Try these (follow the links to check your answer, or look on the last page):

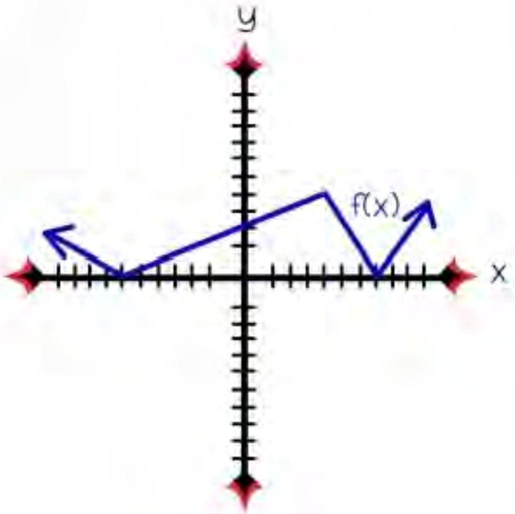
1. $\lim\limits_{x \to 5} x^3 =$

2. $\lim\limits_{x \to 5} x - 7 =$

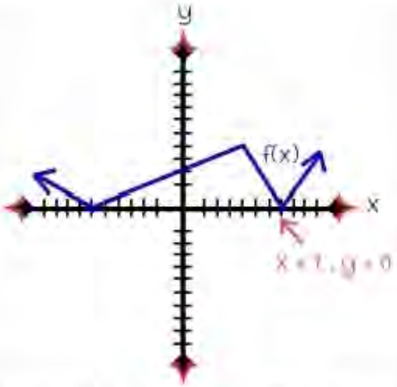
3. $\lim_{x \to -2} 3x + x^2 =$

Check your answers here.

We can also find the limits of functions by looking at graphs. For example, I can use the graph below to find the limit of the function as x approaches 7.



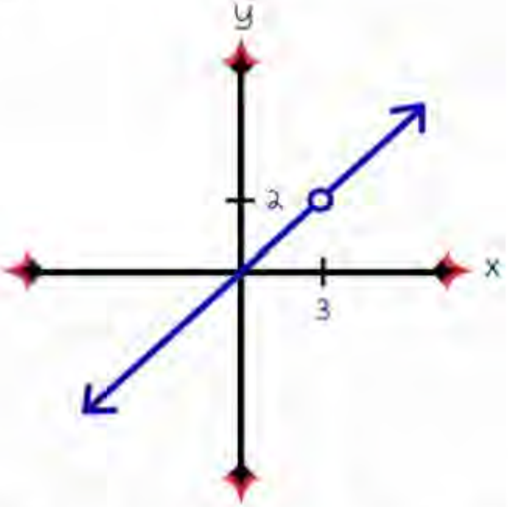
All I need to do is find the place on the x axis where x = 7 and see what the y value of the function is at that point:



So, the limit of the function f(x) when x approaches 7 is 0.

Ok, now some slightly trickier stuff.

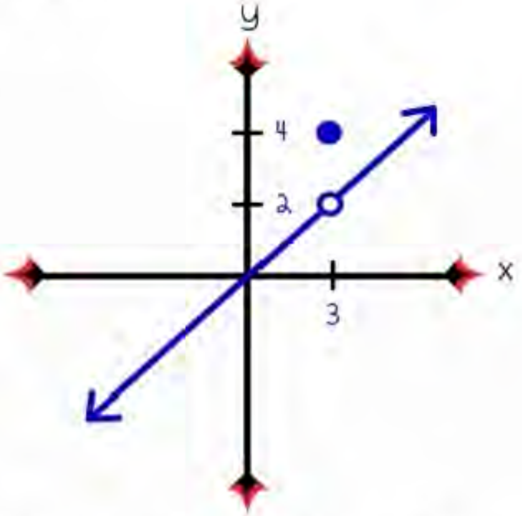
The first thing is that the limit of a function does not depend on what the y value actually is. What we care about is what it **looks** like it's going to be. So, for example, if I have a function with a hole in it like this:



The limit of the function as x approaches 3 is still 2, even though the function isn't defined at x =

3.

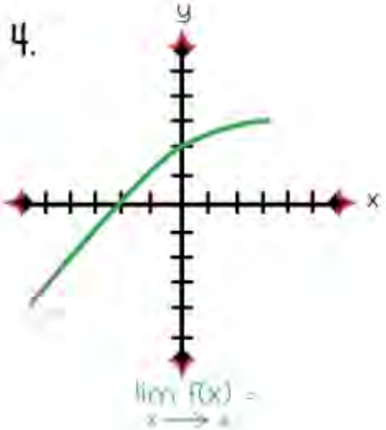
This is true even if the function is actually defined at that point, like in this case:



In the function above, the limit as x approaches 3 is still 2. It doesn't matter that there is actually a point at 3.

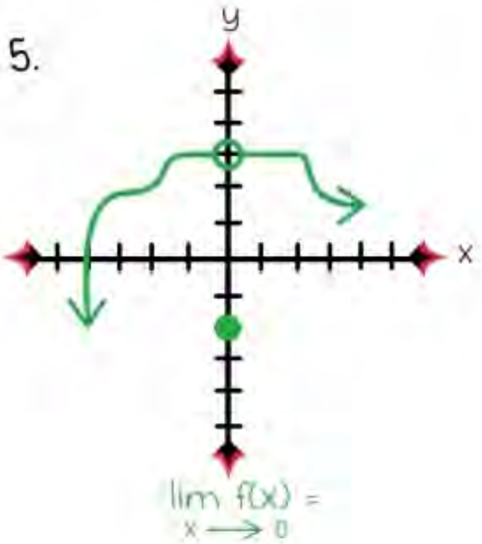
Try These:

4.

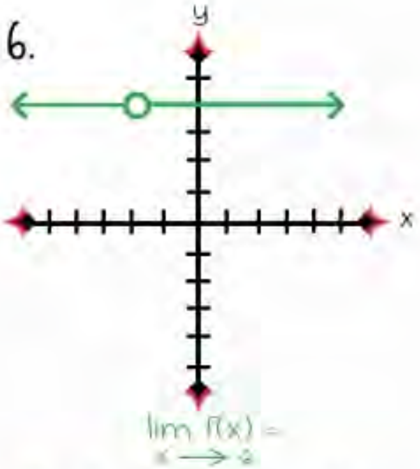


$\lim_{x \to 4} f(x) =$

5.

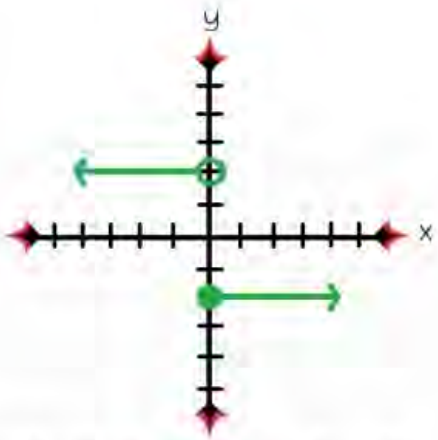


6.



Check your answers here

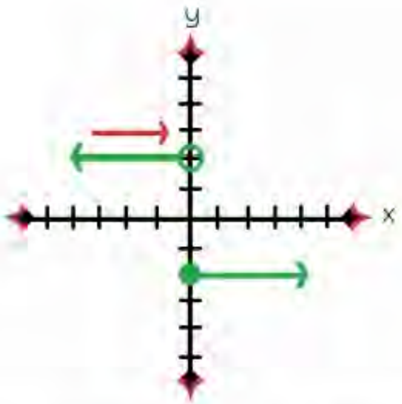
The next complication is that we can take limits of functions from just one side or the other. For example, take a look at this function:



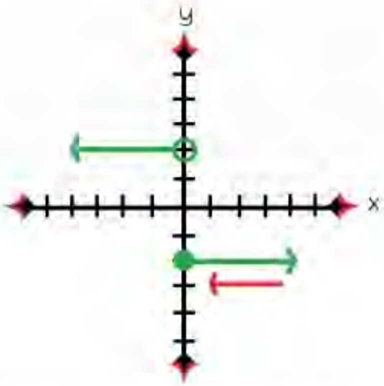
What is the limit as x approaches 0?

It depends on whether we approach it from the left or the right.

If we approach from the left, we would find that the limit is 2:



However, if we approach x = 0 from the right side, we would find that the limit is -2:



If the limit from the left is not the same as the limit from the right, then we say the limit does not exist. In the story, this was the place Kip came to where spider-like creatures were jumping from one path to another, and changing.

However, we can be more specific and just ask about the limit from the left or right. Here is how we write the limit from the left:

$$\lim_{x \to 0^-} f(x) =$$

The little minus sign above the zero means that we're coming from the left side.

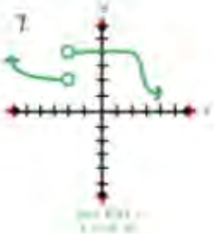
Here is how we write the limit from the right:

$$\lim_{x \to 0^+} f(x) =$$

The little plus sign above the zero means that we're approaching from the right side.

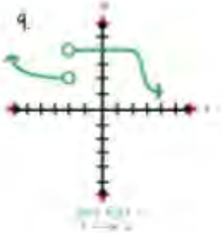
Try these:

7.



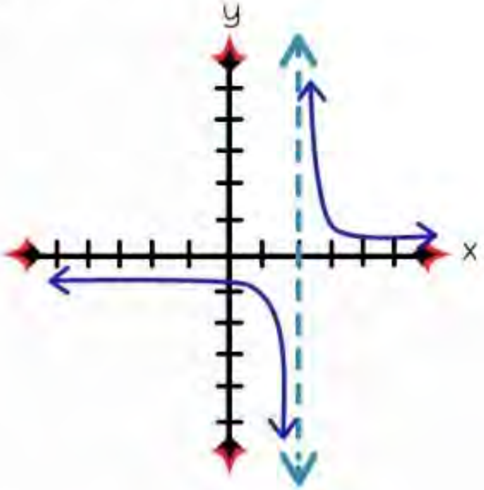
8.

9.



[Check your answers]

This brings us to our next topic: infinite limits. First, let's look at the case when the limit of a function is infinity. This happens when we have a vertical asymptote:



For this function:

$$\lim_{x \to 2^+} f(x) = \infty$$

Because, from the right side, as x approaches 2, the y value gets bigger and bigger and approaches infinity.

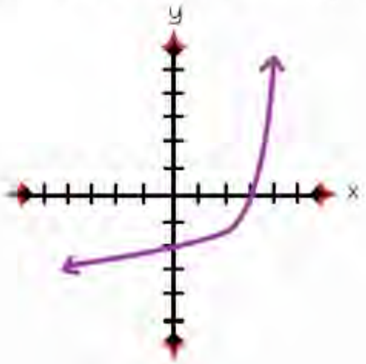
Also,

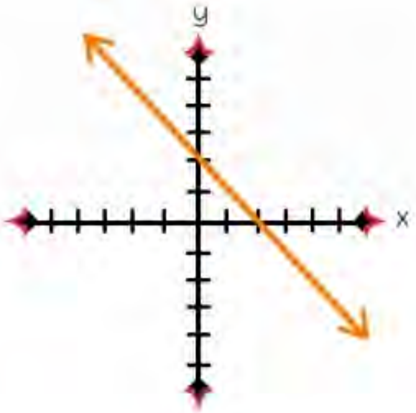
$$\lim_{x \to 2^-} f(x) = -\infty$$

(Since the limit from the left and right are not the same, this also means that the limit as x approaches 2 does not exist.)

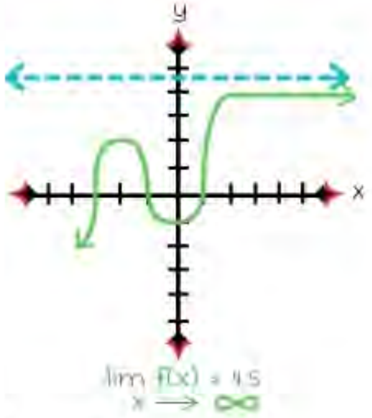
But what about the limit as x approaches infinity?

Most functions just go to infinity when x goes to either positive or negative infinity:

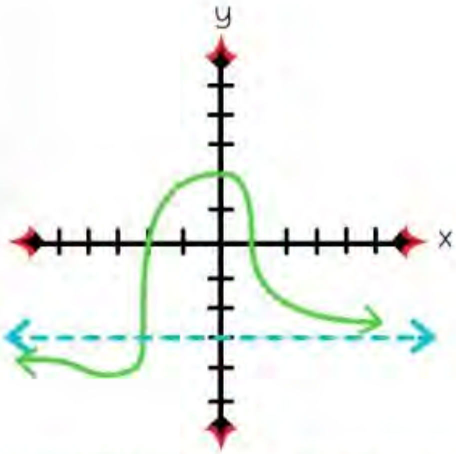




However, some functions have horizontal asymptotes, which means that as x gets really big or really big negatively, the y value approaches some constant value:

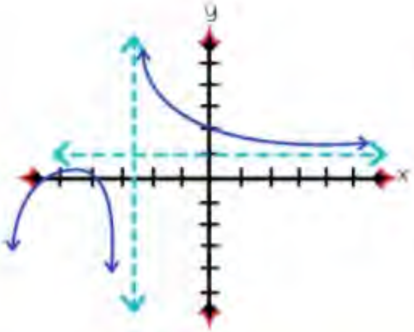


Or, something like this:



$\lim_{x \to -\infty} f(x) = -3$ $\lim_{x \to \infty} f(x) = -3$

Try these:



10. $\lim_{x \to \infty} f(x) =$

11. $\lim_{x \to 3^-} f(x) =$

12. $\lim_{x \to -\infty} f(x) =$

Check your answers

Now let's look at how to do this with equations. At the beginning, I mentioned how with the easy functions you can just plug the x value into the function. However, with some functions this doesn't work because it gives you something undefined. For example,

$$\lim_{x \to 3} \frac{x + 7}{x - 3}$$

If I just plug 3 into my equation, I will get zero on the bottom, which is undefined. In some cases (like in this one) this means I have a vertical asymptote, which means the limit might be positive or negative infinity (or undefined if it goes to positive infinity on one side but negative infinity on the other side.)

In other cases, though, I can do some fancy algebra to get the denominator to cancel out.

For example,

$$\lim_{x \to 3} \frac{x + 4x - 21}{x - 3}$$

Notice that I can factor the numerator:

$$\lim_{x \to 3} \frac{(x + 7)(x - 3)}{x - 3}$$

And now the (x – 3)'s cancel out:

$$\lim_{x \to 3} \frac{(x + 7)\cancel{(x - 3)}}{\cancel{x - 3}} = \lim_{x \to 3} x + 7$$

So now we can just substitute 3 in for x:

$$\lim_{x \to 3} x + 7 = 3 + 7 = 10$$

This is represented in the story by the silver spheres which cancel out the colored lights that are the same on top and bottom.

Try these:

13. $\lim_{x \to 0} \frac{x^3 - 4x^2}{x}$

14. $\lim_{x \to 5} \frac{x^2 - 13x + 40}{x - 5}$

15. $\lim_{x \to -3} \frac{x^2 + 6x + 9}{9 - x^2}$

Check your answers

Another trick for fixing the zero-denominator problem works when you have something plus or minus a square root on the top or bottom:

$$\lim_{x \to 2} \frac{2 - x}{4 - \sqrt{14 + x}}$$

First thing we'll try is plugging 2 in for x:

$$\lim_{x \to 2} \frac{2 - x}{4 - \sqrt{14 + x}}$$

$$\frac{2 - 2}{4 - \sqrt{14 + 2}}$$

$$\frac{0}{4 - \sqrt{16}}$$

$$\frac{0}{4 - 4}$$

$$\frac{0}{0}$$

Since this gives us a zero in the denominator, we have a problem. We can't divide by zero, so we don't know what the value of this limit will be. This is called an indeterminate limit. So, we need to do something to fix the problem of the zero in the denominator.

The trick we use for this is to multiply the top and bottom by the conjugate of the part with the square root.

Conjugate is just a fancy term for "switch the sign in the middle."

For example:

"x + 2" is the conjugate of "x – 2"

Starting over with the original question:

$$\lim_{x \to 2} \frac{2 - x}{4 - \sqrt{14 + x}}$$

I multiply top and bottom by the conjugate of the part with the square root:

$$\lim_{x \to 2} \frac{2 - x}{4 - \sqrt{14 + x}} \cdot \frac{4 + \sqrt{14 + x}}{4 + \sqrt{14 + x}}$$

Why is this a legal math thing to do? Why can we multiply by that red term? The answer is that we're really just multiplying by 1.

For example,

$$\frac{5}{5} = 1 \quad \frac{6}{6} = 1 \quad \frac{x + 5}{x + 5} = 1$$

So, that red part above is actually just a fancy way of writing the number 1. We know that, anytime we multiply a number by 1, that number stays the same, so multiplying by the red part won't change our function.

Here's what happens when we multiply by the conjugate:

$$\lim_{x \to 2} \frac{2 - x}{4 - \sqrt{14 + x}} \cdot \frac{4 + \sqrt{14 + x}}{4 + \sqrt{14 + x}}$$

$$\lim_{x \to 2} \frac{(2 - x)(4 + \sqrt{14 + x})}{(4 - \sqrt{14 + x})(4 + \sqrt{14 + x})}$$

Multiply out (FOIL) the bottom but not the top.

Here's what happens when we multiply out the bottom:

$$(4 - \sqrt{14 + x})(4 + \sqrt{14 + x})$$

$$16 + 4\sqrt{14 + x} - 4\sqrt{14 + x} - (\sqrt{14 + x})^2$$

$$16 + \cancel{4\sqrt{14 + x}} - \cancel{4\sqrt{14 + x}} - (\sqrt{14 + x})^2$$

$$16 - (14 + x)$$

$$2 - x$$

Notice that the two middle terms (called cross terms) cancel out and all that is left is the first term (4) squared minus the last term (the square root of 14 + x) squared. This is what always happens, so you don't really need to go through all the effort of foiling every time.

Here's another example. If I foil this out:

$$(3+\sqrt{x+7})(3-\sqrt{x+7})$$

Notice that the two terms are the same except for the sign in the middle. This is a special factoring case (called "difference of squares.")

When we multiply these out we get this:

$$3^2-(X+7)$$

(The first term squared minus the second term squared.)

But, let's get back to the problem we were working on:

$$\lim_{x \to 2} \frac{2-X}{4-\sqrt{14+X}}$$

We have simplified it now so that it looks like this:

$$\lim_{x \to 2} \frac{(2-x)(4+\sqrt{14+X})}{2-X}$$

The 2 - x cancels on the top and on the bottom:

$$\lim_{x \to 2} \frac{\cancel{(2-x)}4+\sqrt{14+x}}{\cancel{2-x}}$$

$$\lim_{x \to 2} 4+\sqrt{14+x}$$

The last step is just to plug the 2 in for x:

$$4+\sqrt{14+2}$$

$$4+4$$

$$8$$

And, that's our answer, the limit of the function as x approaches 2 is 8.

Try these:

16. $\lim_{x \to 2} \frac{2-x}{5-\sqrt{23+x}}$

17. $\lim_{x \to 7} \frac{x^2-4x-21}{\sqrt{x+57}-8}$

18. $\lim_{x \to -3} \frac{x+3}{5-\sqrt{x+28}}$

Check your answers

Ok, the last thing we'll cover here is determining the limit as x goes to positive or negative infinity from an equation. (Which is what the scale did in the story.)

To do that, we just look at the highest powers of x on the top and bottom of a rational expression.

For example:

$$\lim_{x \to \infty} \frac{x^2 + 2x + 9}{x - 3}$$

Here, the highest power of x on the top is 2, and the highest power on the bottom is 1.

If the greatest power of x on the top is greater than the greatest power of x on the bottom, then there is no limit as x goes to infinity.

If the power on the bottom is greater than the power on the top, like this:

$$\lim_{x \to \infty} \frac{x^2 + 2x + 9}{x^5 - 3}$$

Then the limit is zero. (Which makes sense, because as x gets really big, the bottom of that function is waaay bigger than the top function. A number divided by a number that much bigger is so small it's close to zero.)

If the greatest powers of x are the same on top and bottom:

$$\lim_{x \to \infty} \frac{x^2 + 2x + 9}{x^2 - 3}$$

Then the limit is the ratio of the coefficients of the greatest powers of x. So, in the above example the coefficients of the x squared terms are both 1, so the limit as x goes to infinity is 1.

Here's another example:

$$\lim_{x \to \infty} \frac{2x^2 - 5x}{3x^2 + 7x}$$

The limit as x goes to infinity here is 2/3.

Try these:

19. $$\lim_{x \to \infty} \frac{x^2 - 8x}{3x^5 - 12x + 100}$$

20. $$\lim_{x \to \infty} \frac{5 - 8x^2}{2x^2 + 3}$$

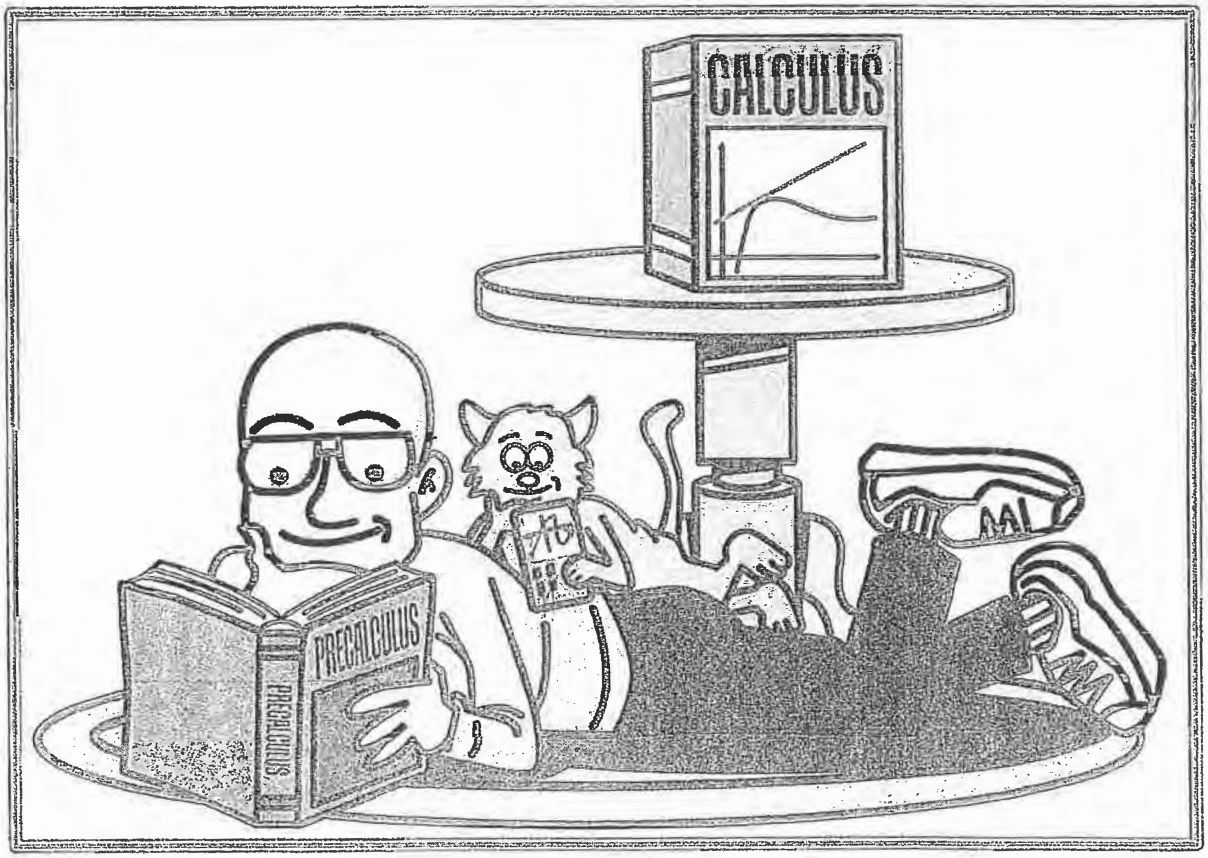
21. $$\lim_{x \to \infty} \frac{55 - x^4 + 3x^2}{2 + 4x - 8x^3}$$

Check your answers

Name: ____________________ Class: __________ Date: __________ Score: __________

RU Ready for Some Calculus?

A Precalculus Review



Student Version

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

RU for Some Calculus? A Precalculus Review

Table of Contents:

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

To the Student:

Picture a block of Swiss cheese. It is filled with holes and yet it stays in one piece. But this block has to be cut into slices. If there are too many holes, the slice will simply fall apart.

So it is with calculus. The AP Calculus course you are about to take is based on your foundation in mathematics – all the math that you have ever learned will come into play in this course. If you are taking calculus, it is possible that some of that material you knew fairly well at one time, but unfortunately, without everyday use, you just plain forget it. It is also possible that you never really learned it at all.

When you start your AP Calculus course, teachers make the assumption that you have mastered a lot of mathematics and techniques that you need to know are part of you. But it is a bad assumption and worse, a lot of teachers know it. In the past, teachers would start a new year by reviewing and getting everyone's knowledge at the same level. But in calculus, there is simply not enough time to spend time in review

So teachers teach AP Calculus knowing that there are extreme deficiencies in their student's math skills. And if the deficiencies are serious, the entire year crumbles like a piece of Swiss cheese with too many holes – holes in mathematical knowledge!

So what is the answer? Review all the math you have ever had? No, that just takes too long and who really cares enough to do that.

So this booklet contains all the material from precalculus that you really need to know going into AP calculus. It does not necessarily review the most difficult concepts of precalculus, but it takes the concepts that were in precalculus and are quite likely to show up in AP Calculus and teaches you, once and for all, to handle problems using those concepts.

For instance, the concept of complex fractions, fractions within fractions, usually only show up in precalculus when you are studying that concept. They rarely show up in word problems or in any other context. So you learn them when you need them, and you forget them 10 minutes after the test.

But in AP calculus, complex fractions occur fairly frequently Calculus is hard enough and if you lose points on a problem, you want it to be because you had a conceptual issue with the calculus topic, not because your knowledge of precalculus, specifically complex fractions, was faulty.

So this booklet contains just those concepts that are important for you in learning AP calculus. Topics like the conic sections, imaginary numbers, and finding rational zeros of functions, while important in precalculus, are rarely used in AP Calculus so they aren't included in this booklet.

You can be sure that if you review and master all the topics in this booklet, you are well on your way to doing well in AP Calculus. The reason is that many students worldwide struggle in AP calculus because their precalculus abilities are not good. Spending about eight hours on this booklet in total insures that is not going to happen to you!

RU Ready?

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

Let's talk about your calculus course. You are taking an Advanced Placement Calculus course. It is either AB Calculus or BC Calculus. Let's understand what these unusual names mean. The A.P Calculus program started in the year 1956. There was only one calculus exam given in these early years and it was called "Math." However, once the AP Calculus program got rolling fully, the courses were split into AB and BC and the first year there was a specific AB and BC exam was in the year 1969 There were three general topics into which all math problems fall

A Topics: these are precalculus concepts. They use no calculus but are considered necessary to understand and master before a student can master calculus.

B Topics. these are comprised of the calculus concepts taught in a first-year college calculus course.

C Topics: these are the calculus concepts taught in a second-year college calculus course.

So in a typical AB Calculus course, students will see problems including A topics and B topics and while in a BC course, students will see problems including B topics and C topics. Before the year 2000, there were problems on the AB exam that were strictly A topics no calculus was required. That is no longer true. In reality, all 45 multiple-choice questions and 6 free response questions on the AB exam are B topic questions. They are designed to test calculus.

So, although A topics are not specifically tested, students still need to understand them. You need to be able to solve equations, add algebraic fractions, find logarithms, and find trig functions of special angles. As with spelling, while students are not tested specifically on their spelling abilities by the time they get to high school, it is assumed that they know how to spell.

So, as a review, I have chosen 21 precalculus (A topics) that you really need to know and have mastered before you start your calculus book. This is not meant to be a complete review and if some of these topics are still a mystery to you, ask your teacher for an algebra, trigonometry, or precalculus book to borrow to sharpen your skills. The topics are not the only ones essential to mastering precalculus but were chosen because they crop up continuously in calculus examples. The way you see these examples expressed demonstrate how you will see them in calculus problems.

After every general topic and description, you will see sample problems with solutions worked out. On the back of each page, you will find roughly 12-15 problems that are similar to the examples. Your teachers might assign these over the summer It is suggested that you do one topic a day Your teacher might give you a 25-question multiple choice test the first day or so in class or give it to you as a summer take-home type exam. Please take it seriously Do well in this and you have mastered all the precalculus you need for AP Calculus. You will feel good that your block of Swiss cheese has few holes and when the block is sliced (and you break down calculus concepts), that the piece will stay together

Best of luck.

RU Ready?

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

A. Functions

The lifeblood of precalculus is functions. A **function** is a set of points (x,y) such that for every x, there is one and only one y In short, in a function, the x-values cannot repeat while the y-values can. In AB Calculus, all of your graphs will come from functions.

The notation for functions is either "$y =$" or "$f(x)=$" In the $f(x)$ notation, we are stating a rule to find y given a value of x.

1 If $f(x)=x^2-5x+8$, find a) $f(-6)$ b) $f\left(\frac{3}{2}\right)$ c) $\frac{f(x+h)-f(x)}{h}$

$$\boxed{\begin{array}{l} \text{a) } f(-6)=(-6)^2-5(-6)+8 \\ 36+30+8 \\ 74 \end{array}}$$

$$\boxed{\begin{array}{l} \text{b) } f\left(\frac{3}{2}\right)=\left(\frac{3}{2}\right)^2-5\left(\frac{3}{2}\right)+8 \\ \frac{9}{4}-\frac{15}{2}+8 \\ \frac{11}{4} \end{array}}$$

$$\boxed{\begin{array}{l} \text{c) } \frac{f(x+h)-f(x)}{h} \\ \frac{(x+h)^2-5(x+h)+8-(x^2-5x+8)}{h} \\ \frac{x^2+2xh+h^2-5x-5h+8-x^2+5x-8}{5} \\ \frac{h^2+2xh-5h}{h}=\frac{h(h+2x-5)}{h}=h+2x-5 \end{array}}$$

Functions do not always use the variable x. In calculus, other variables are used liberally

2. If $A(r)=\pi r^2$, find a) $A(3)$ b) $A(2s)$ c) $A(r+1)-A(r)$

$$\boxed{A(3)=9\pi}$$

$$\boxed{A(2s)=\pi(2s)^2=4\pi s^2}$$

$$\boxed{\begin{array}{l} A(r+1)-A(r)=\pi(r+1)^2-\pi r^2 \\ \pi(2r+1) \end{array}}$$

One concept that comes up in AP calculus is **composition of functions** The format of a composition of functions is. plug a value into one function, determine an answer, and plug that answer into a second function.

3 If $f(x)=x^2-x+1$ and $g(x)=2x-1$, a) find $f(g(-1))$ b) find $g(f(-1))$ c) show that $f(g(x))\neq g(f(x))$

$$\boxed{\begin{array}{l} g(-1)=2(-1)-1=-3 \\ f(-3)=9+3+1=13 \end{array}}$$

$$\boxed{\begin{array}{l} f(-1)=1+1+1=3 \\ g(3)=2(3)-1=5 \end{array}}$$

$$\boxed{\begin{array}{l} f(g(x))=f(2x-1)=(2x-1)^2-(2x-1)+1 \\ \quad =4x^2-4x+1-2x+1+1=4x^2-6x+3 \\ g(f(x))=g(x^2-x+1)=2(x^2-x+1)-1 \\ \quad =2x^2-2x+1 \end{array}}$$

Finally, expect to use **piecewise functions**. A piecewise function gives different rules, based on the value of x

4 If $f(x)=\begin{cases} x^2-3, & x\geq 0 \\ 2x+1, & x<0 \end{cases}$, find a) $f(5)$ b) $f(2)-f(-1)$ c) $f(f(1))$

$$\boxed{f(5)=25-3=22}$$

$$\boxed{f(2)-f(-1)=1-(-1)=2}$$

$$\boxed{f(1)=-2,\ f(-2)=-3}$$

ODD PROBLEMS ONLY

Name: ______________________ Class: ___________ Date: ___________ Score: ___________

A. Function Assignment

- If $f(x) = 4x - x^2$, find

1 $f(4) - f(-4)$

2. $\sqrt{f\left(\frac{3}{2}\right)}$

3 $\frac{f(x+h) - f(x-h)}{2h}$

- If $V(r) = \frac{4}{3}\pi r^3$, find

4 $V\left(\frac{3}{4}\right)$

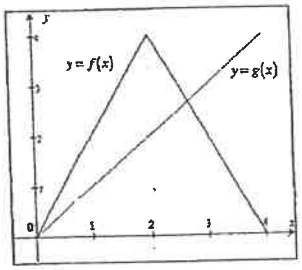
5 $V(r+1) - V(r-1)$

6 $\frac{V(2r)}{V(r)}$

- If $f(x)$ and $g(x)$ are given in the graph below, find

7. $(f-g)(3)$

8. $f(g(3))$



- If $f(x) = x^2 - 5x + 3$ and $g(x) = 1 - 2x$, find

9 $f(g(x))$

- If $f(x) = \begin{cases} \sqrt{x+2} - 2, & x \geq 2 \\ x^2 - 1, & 0 \leq x < 2 \\ -x, & x < 0 \end{cases}$, find

10. $f(0) - f(2)$

11 $\sqrt{5 - f(-4)}$

12. $f(f(3))$

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

B. Domain and Range

First, since questions in calculus usually ask about behavior of functions in intervals, understand that intervals can be written with a description in terms of $<, \leq, >, \geq$ or by using **interval notation**.

Description	Interval notation	Description	Interval notation	Description	Interval notation
$x>a$	(a,∞)	$x\leq a$	$(-\infty,a]$	$a\leq x<b$	$[a,b)$
$x\geq a$	$[a,\infty)$	$a<x<b$	(a,b) · open interval	$a<x\leq b$	$(a,b]$
$x<a$	$(-\infty,a)$	$a\leq x\leq b$	$[a,b]$ · closed interval	All real numbers	$(-\infty,\infty)$

If a solution is in one interval or the other, interval notation will use the connector $\cup$ So $x\leq 2$ or $x>6$ would be written $(-\infty,2]\cup(6,\infty)$ in interval notation. Solutions in intervals are usually written in the easiest way to define it. For instance, saying that $x<0$ or $x>0$ or $(-\infty,0)\cup(0,\infty)$ is best expressed as $x\neq 0$

The **domain of a function** is the set of allowable x-values. The domain of a function f is $(-\infty,\infty)$ except for values of x which create a zero in the denominator, an even root of a negative number or a logarithm of a non-positive number The domain of $a^{p(x)}$ where a is a positive constant and $p(x)$ is a polynomial is $(-\infty,\infty)$

- Find the domain of the following functions using interval notation.

1 $f(x)=x^2-4x+4$ $\boxed{(-\infty,\infty)}$

2. $y=\dfrac{6}{x-6}$ $\boxed{x\neq 6}$

3 $y=\dfrac{2x}{x^2-2x-3}$ $\boxed{x\neq -1, x\neq 3}$

4 $y=\sqrt{x+5}$ $\boxed{[-5,\infty)}$

5 $y=\sqrt[3]{x+5}$ $\boxed{(-\infty,\infty)}$

6. $y=\dfrac{x^2+4x+6}{\sqrt{2x+4}}$ $\boxed{(-2,\infty)}$

The **range of a function** is the set of allowable y-values. Finding the range of functions algebraically isn't as easy (it really is a calculus problem), but visually, it is the [lowest possible y-value, highest possible y-value]. Finding the range of some functions are fairly simple to find if you realize that the range of $y=x^2$ is $[0,\infty)$ as any positive number squared is positive. Also the range of $y=\sqrt{x}$ is also positive as the domain is $[0,\infty)$ and the square root of any positive number is positive. The range of $y=a^x$ where a is a positive constant is $(0,\infty)$ as constants to powers must be positive.

- Find the range of the following functions using interval notation.

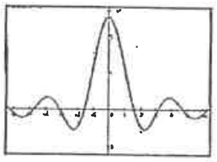
7 $y=1-x^2$ $\boxed{(-\infty,1]}$

8 $y=\dfrac{1}{x^2}$ $\boxed{(0,\infty)}$

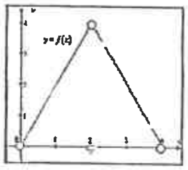
9 $y=\sqrt{x-8}+2$ $\boxed{[2,\infty)}$

- Find the domain and range of the following functions using interval notation.

10



Domain: $(-\infty,\infty)$
Range: $[-0.5,2.5]$



11

Domain: $(0,4)$
Range: $[0,4)$

RU Ready?

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

B. Domain and Range Assignment

• Find the domain of the following functions using interval notation.

1 $f(x)=3$

2. $y=x^3-x^2+x$

3 $y=\dfrac{x^3-x^2+x}{x}$

4 $y=\dfrac{x-4}{x^2-16}$

5 $f(x)=\dfrac{1}{4x^2-4x-3}$

6 $y=\sqrt{2x-9}$

7 $f(t)=\sqrt{t^3+1}$

8. $f(x)=\sqrt[5]{x^2-x-2}$

9 $y=5^{x^2-4x-2}$

10 $y=\log(x-10)$

11 $y=\dfrac{\sqrt{2x+14}}{x^2-49}$

12. $y=\dfrac{\sqrt{5-x}}{\log x}$

Find the range of the following functions.

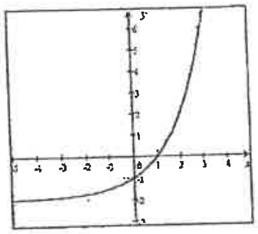
13 $y=x^4+x^2-1$

14 $y=100^x$

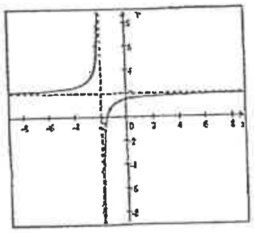
15 $y=\sqrt{x^2+1}+1$

Find the domain and range of the following functions using interval notation.

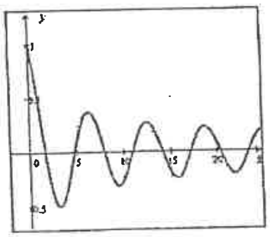
16



17



18

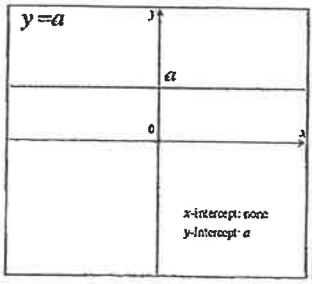


ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

C. Graphs of Common Functions

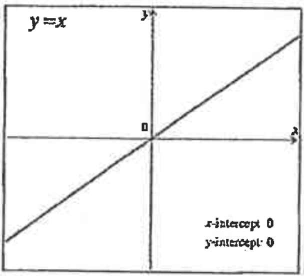
There are certain graphs that occur all the time in calculus and students should know the general shape of them, where they hit the x-axis (zeros) and y-axis (y-intercept), as well as the domain and range. There are no assignment problems for this section other than students memorizing the shape of all of these functions. In section 5, we will talk about transforming these graphs.



Function. $y=a$

Domain. $(-\infty,\infty)$

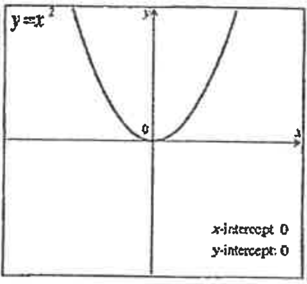
Range: $[a,a]$



Function. $y=x$

Domain. $(-\infty,\infty)$

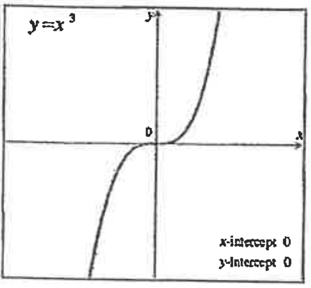
Range: $(-\infty,\infty)$



Function. $y=x^2$

Domain. $(-\infty,\infty)$

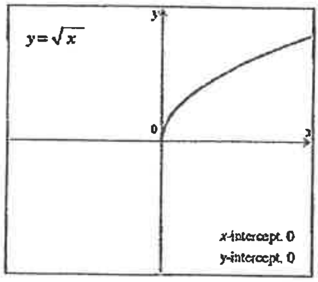
Range: $[0,\infty)$



Function. $y=x^3$

Domain $(-\infty,\infty)$

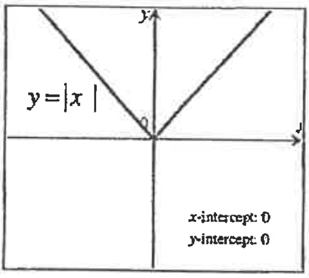
Range: $(-\infty,\infty)$



Function: $y=\sqrt{x}$

Domain. $[0,\infty)$

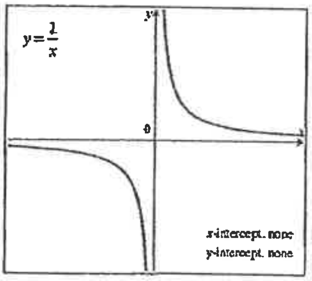
Range: $[0,\infty)$



Function. $y=|x|$

Domain. $(-\infty,\infty)$

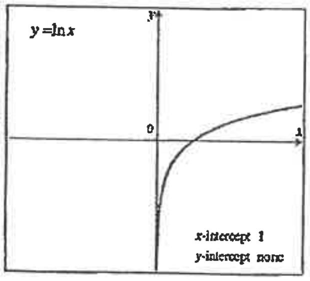
Range: $[0,\infty)$



Function $y=\dfrac{1}{x}$

Domain. $x\neq 0$

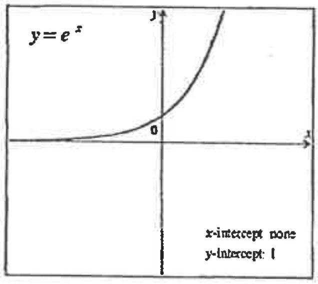
Range: $y\neq 0$



Function: $y=\ln x$

Domain. $(0,\infty)$

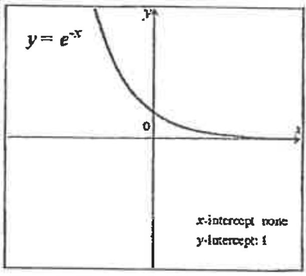
Range: $(-\infty,\infty)$



Function: $y=e^x$

Domain. $(-\infty,\infty)$

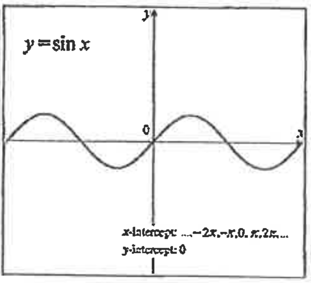
Range: $(0,\infty)$



Function. $y=e^{-x}$

Domain. $(-\infty,\infty)$

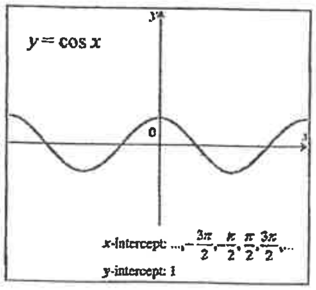
Range. $(0,\infty)$



Function: $y=\sin x$

Domain. $(-\infty,\infty)$

Range: $[-1,1]$



Function. $y=\cos x$

Domain: $(-\infty,\infty)$

Range: $[-1,1]$

Name: ____________________ Class: __________ Date: __________ Score: __________

C. Graphs of Common Functions - Assignment

Sketch each of the following as accurately as possible. You will need to be VERY familiar with each of these graphs throughout the year You may use a graphing calculator if you have access to one over the summer On the first day of class, you will all have a TI-84 to use and could therefore finish this part at that time. Another option is to find a graphing app (there are free ones) or generate a table of values on your scientific calculator Again these are VERY important graphs to know Be very accurate with regards to "open circles" and "closed circles."

1. $y = x$

2. $y = x^2$

3. $y = x^3$

4 $y = \sqrt{x}$

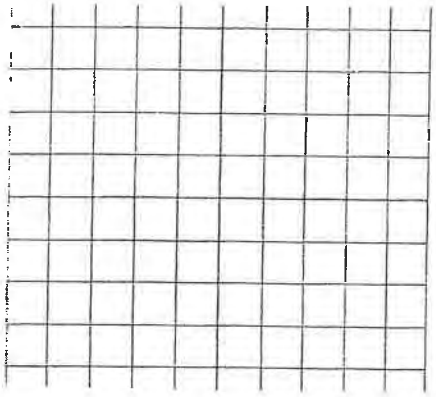
5. $y = |x|$

6 $y = \frac{|x|}{x}$

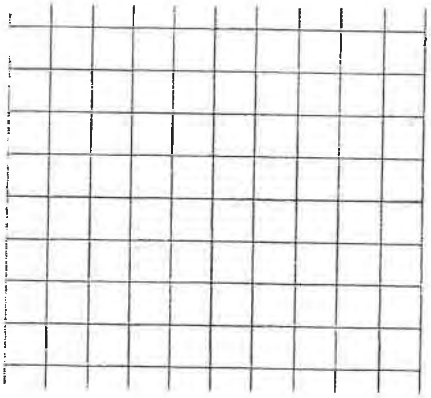
ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

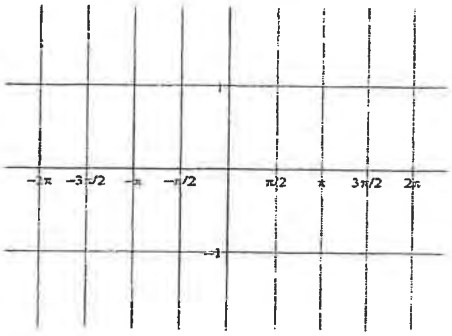
7 $y = x^{1/3}$



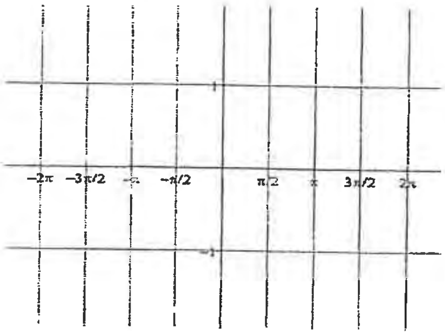
8. $y = x^{2/3}$



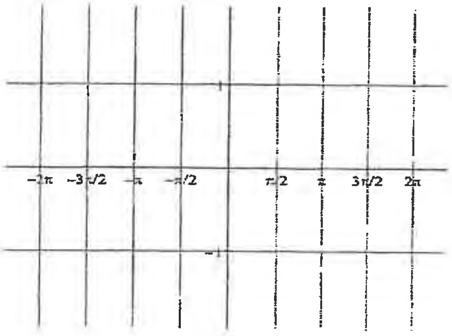
9 $y = \sin x$



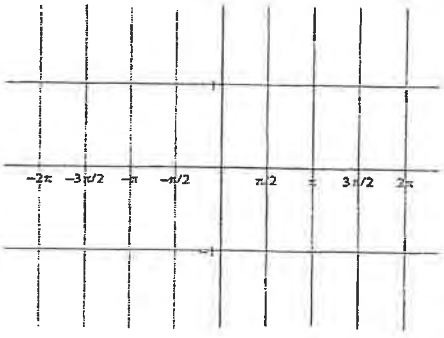
10. $y = \cos x$



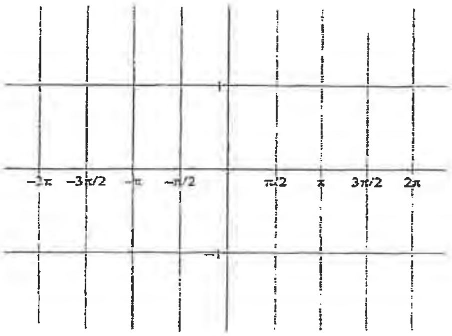
11 $y = \tan x$



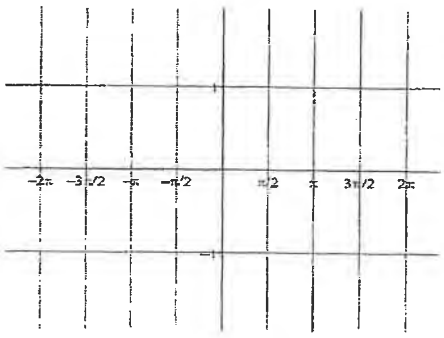
12. $y = \cot x$



13 $y = \sec x$



14. $y = \csc x$



ODD PROBLEMS ONLY

Name: ______________________ Class: ___________ Date: ___________ Score: ___________

15 $y = e^x$

16 $y = \ln x$

17 $y = \frac{1}{x}$

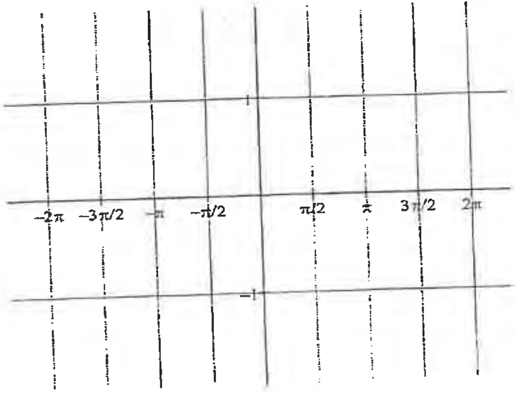
18 $y = [\![x]\!]$

19 $y = \frac{1}{x^2}$

20. $y = 2^x$

21 $y = \sqrt{4 - x^2}$

22. $y = \frac{\sin x}{x}$



ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

D. Even and Odd Functions

Functions that are even have the characteristic that for all a, $f(-a)=f(a)$ What this says is that plugging in a positive number a into the function or a negative number $-a$ into the function makes no difference you will get the same result. Even functions are symmetric to the y-axis.

Functions that are odd have the characteristic that for all a, $f(-a)=-f(a)$ What this says is that plugging in a negative number $-a$ into the function will give you the same result as plugging in the positive number and taking the negative of that. So, odd functions are symmetric to the origin. If a graph is symmetric to the x-axis, it is not a function because it fails the vertical-line test.

1 Of the common functions in section 3, which are even, which are odd, and which are neither?

$$\boxed{\begin{array}{l}\text{Even: } y=a,\ y=x^2,\ y=|x|,\ y=\cos x \qquad \text{Odd: } y=x,\ y=x^3,\ y=\dfrac{1}{x},\ y=\sin x \\ \text{Neither: } y=\sqrt{x},\ y=\ln x,\ y=e^x,\ y=e^{-x}\end{array}}$$

2. Show that the following functions are even.

a) $f(x)=x^4-x^2+1$

$$\boxed{\begin{array}{l}f(-x)=(-x)^4-(-x)^2+1 \\ =x^4-x^2+1=f(x)\end{array}}$$

b) $f(x)=\left|\dfrac{1}{x}\right|$

$$\boxed{f(-x)=\left|\dfrac{1}{-x}\right|=\left|\dfrac{1}{x}\right|=f(x)}$$

c) $f(x)=x^{2/3}$

$$\boxed{\begin{array}{l}f(-x)=(-x)^{2/3}=\left(\sqrt[3]{-x}\right)^2 \\ =\left(\sqrt[3]{x}\right)^2=f(x)\end{array}}$$

3 Show that the following functions are odd.

a) $f(x)=x^3-x$

$$\boxed{\begin{array}{l}f(-x)=(-x)^3+x \\ =x-x^3=-f(x)\end{array}}$$

b) $f(x)=\sqrt[3]{x}$

$$\boxed{f(-x)=\sqrt[3]{-x}=-\sqrt[3]{x}=-f(x)}$$

c) $f(x)=e^x-e^{-x}$

$$\boxed{f(-x)=e^{-x}-e^{x}=-\left(e^x-e^{-x}\right)=-f(x)}$$

4. Determine if $f(x)=x^3-x^2+x-1$ is even, odd, or neither Justify your answer

$$\boxed{f(-x)=-x^3-x^2-x-1\neq f(x) \text{ so } f \text{ is not even.} \qquad -f(x)=-x^3+x^2-x-1\neq f(-x) \text{ so } f \text{ is not odd.}}$$

Graphs may not be functions and yet have x-axis or y-axis or both. Equations for these graphs are usually expressed in "implicit form" where it is not expressed as "$y=$" or "$f(x)=$" If the equation does not change after making the following replacements, the graph has these symmetries:

x-axis: y with $-y$ y-axis: x with $-x$ origin: both x with $-x$ and y with $-y$

5 Determine the symmetry for $x^2+xy+y^2=0$

$$\boxed{\begin{array}{l}x-\text{axis: } x^2+x(-y)+(-y)^2=0\Rightarrow x^2-xy+y^2=0 \text{ so not symmetric to } x-\text{axis} \\ y-\text{axis: } (-x)^2+(-x)(y)+y^2=0\Rightarrow x^2-xy+y^2=0 \text{ so not symmetric to } y-\text{axis} \\ \text{origin: } (-x)^2+(-x)(-y)+y^2=0\Rightarrow x^2+xy+y^2=0 \text{ so symmetric to origin}\end{array}}$$

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

D. Even and odd functions - Assignment

- Show work to determine if the following functions are even, odd, or neither

1. $f(x)=7$

2. $f(x)=2x^2-4x$

3. $f(x)=-3x^3-2x$

4 $f(x)=\sqrt{x+1}$

5. $f(x)=\sqrt{x^2+1}$

6. $f(x)=8x$

7. $f(x)=8x-\frac{1}{8x}$

8 $f(x)=|8x|$

9. $f(x)=|8x|-8x$

Show work to determine if the graphs of these equations are symmetric to the *x*-axis, *y*-axis or the origin.

10. $4x=1$

11. $y^2=2x^4+6$

12. $3x^2=4y^3$

13. $x=|y|$

14. $|x|=|y|$

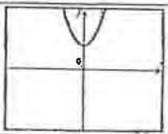
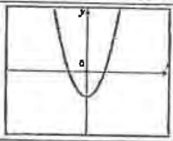
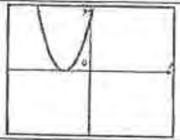
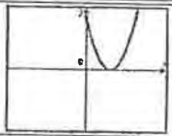
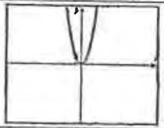
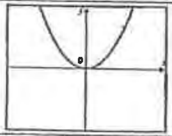
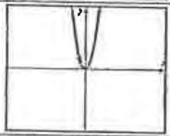
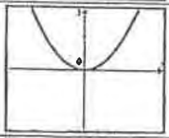
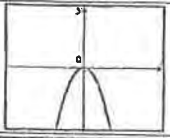
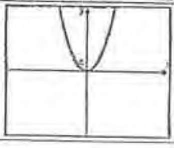
15. $|x|=y^2+2y+1$

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

E. Transformation of Graphs

A curve in the form $y=f(x)$, which is one of the basic common functions from section C can be transformed in a variety of ways. The shape of the resulting curve stays the same but zeros and y-intercepts might change and the graph could be reversed. The table below describes transformations to a general function $y=f(x)$ with the parabolic function $f(x)=x^2$ as an example.

Notation	How $f(x)$ changes	Example with $f(x)=x^2$
$f(x)+a$	Moves graph up a units	
$f(x)-a$	Moves graph down a units	
$f(x+a)$	Moves graph a units left	
$f(x-a)$	Moves graph a units right	
$a\ f(x)$	$a>1$ Vertical Stretch	
$a\ f(x)$	$0<a<1$ Vertical shrink	
$f(ax)$	$a>1$ Horizontal compress (same effect as vertical stretch)	
$f(ax)$	$0<a<1$ Horizontal elongated (same effect as vertical shrink)	
$-f(x)$	Reflection across x-axis	
$f(-x)$	Reflection across y-axis	

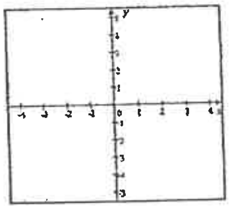
ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

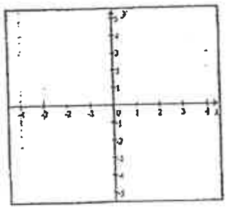
E. Transformation of Graphs Assignment

- Sketch the following equations:

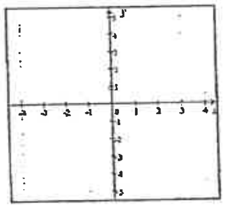
1 $y=-x^2$



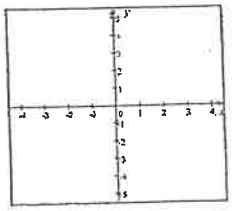
2. $y=2x^2$



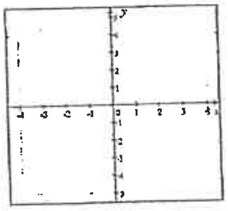
3. $y=(x-2)^2$



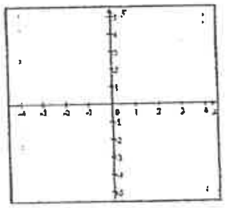
4 $y=2-\sqrt{x}$



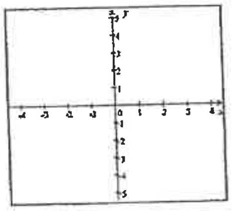
5 $y=\sqrt{x+1}+1$



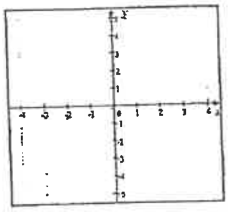
6. $y=\sqrt{4x}$



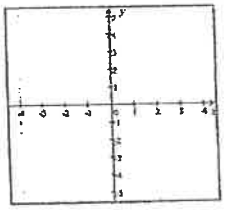
7 $y=|x+1|-3$



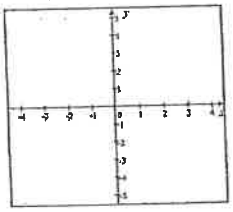
8. $y=-2|x-1|+4$



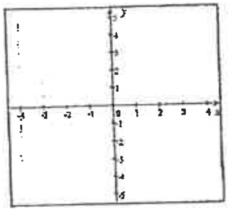
9. $y=-\left|\frac{x}{2}\right|-1$



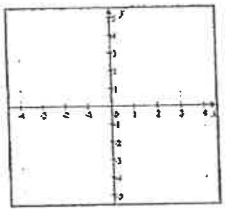
10. $y=2^x-2$



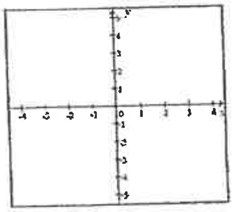
11 $y=-2^{x+2}$



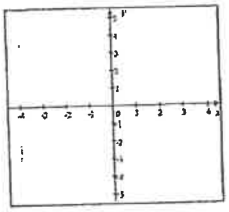
12. $y=2^{-2x}$



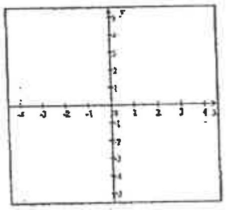
13 $y=\frac{1}{x-2}$



14. $y=\frac{-2}{x+1}$



15 $y=\frac{1}{(x+2)^2}-3$



ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

F. Special Factorization

While factoring skills were more important in the days when A topics were specifically tested, students still must know how to factor The special forms that occur most regularly are:

Common factor: $x^3+x^2+x=x\left(x^2+x+1\right)$

Difference of squares: $x^2-y^2=(x+y)(x-y)$ or $x^{2n}-y^{2n}=\left(x^n+y^n\right)\left(x^n-y^n\right)$

Perfect squares: $x^2+2xy+y^2=(x+y)^2$

Perfect squares: $x^2-2xy+y^2=(x-y)^2$

Sum of cubes. $x^3+y^3=(x+y)\left(x^2-xy+y^2\right)$ - Trinomial unfactorable

Difference of cubes: $x^3-y^3=(x-y)\left(x^2+xy+y^2\right)$ - Trinomial unfactorable

Grouping: $xy+xb+ay+ab=x(y+b)+a(y+b)=(x+a)(y+b)$

The term "factoring" usually means that coefficients are rational numbers. For instance, x^2-2 could technically be factored as $\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)$ but since $\sqrt{2}$ is not rational, we say that x^2-2 is not factorable. It is important to know that x^2+y^2 is unfactorable.

- Completely factor the following expressions.

1 $4a^2+2a$ $\boxed{2a(a+2)}$

2. $x^2+16x+64$ $\boxed{(x+8)^2}$

3 $4x^2-64$ $\boxed{4(x+4)(x-4)}$

4. $5x^4-5y^4$ $\boxed{5\left(x^2+1\right)(x+1)(x-1)}$

5 $16x^2-8x+1$ $\boxed{(4x-1)^2}$

6. $9a^4-a^2b^2$ $\boxed{a^2(3a+b)(3a-b)}$

7 $2x^2-40x+200$ $\boxed{2(x-10)^2}$

8. x^3-8 $\boxed{(x-2)\left(x^2+2x+4\right)}$

9 $8x^3+27y^3$ $\boxed{(2x+3y)\left(4x^2-6xy+9y^2\right)}$

10 x^4+11x^2-80 $\boxed{(x+4)(x-4)\left(x^2+5\right)}$

11 x^4-10x^2+9 $\boxed{(x+1)(x-1)(x+3)(x-3)}$

12. $36x^2-64$ $\boxed{4(3x+4)(3x-4)}$

13 x^3-x^2+3x-3 $\boxed{\begin{array}{l} x^2(x-1)+3(x-1) \\ (x-1)\left(x^2+3\right) \end{array}}$

14 $x^3+5x^2-4x-20$ $\boxed{\begin{array}{l} x^2(x+5)-4(x+5) \\ (x+5)(x-2)(x+2) \end{array}}$

15 $9-\left(x^2+2xy+y^2\right)$ $\boxed{\begin{array}{l} 9-(x+y)^2 \\ (3+x+y)(3-x-y) \end{array}}$

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

F. Special Factorization - Assignment

- Completely factor the following expressions

1 $x^3 - 25x$

2. $30x - 9x^2 - 25$

3 $3x^2 - 5x^2 + 2x$

4 $3x^8 - 3$

5 $16x^4 - 24x^2y + 9y^2$

6. $9a^4 - a^2b^2$

7 $4x^4 + 7x^2 - 36$

8 $250x^3 - 128$

9 $\frac{8x^3}{125} + \frac{64}{y^3}$

10. $x^5 + 17x^3 + 16x$

11 $144 + 32x^2 - x^4$

12. $16x^{4a} - y^{8a}$

13 $x^3 - xy^2 + x^2y - y^3$

14 $x^6 - 9x^4 - 81x^2 + 729$

15 $x^2 - 8xy + 16y^2 - 25$

16 $x^5 + x^3 + x^2 + 1$

17 $x^6 - 1$

18. $x^6 + 1$

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

G. Linear Functions

Probably the most important concept from precalculus that is required for differential calculus is that of linear functions. The formulas you need to know backwards and forwards are:

Slope: Given two points (x_1,y_1) and (x_2,y_2), the slope of the line passing through the points can be written as:

$$m=\frac{\text{rise}}{\text{run}}=\frac{\Delta y}{\Delta x}=\frac{y_2-y_1}{x_2-x_1}$$

Slope intercept form the equation of a line with slope m and y-intercept b is given by $y=mx+b$

Point-slope form: the equation of a line passing through the points (x_1,y_1) and slope m is given by $y-y_1=m(x-x_1)$ While you might have preferred the simplicity of the $y=mx+b$ form in your algebra course, the $y-y_1=m(x-x_1)$ form is far more useful in calculus.

Intercept form: the equation of a line with x-intercept a and y-intercept b is given by $\frac{x}{a}+\frac{y}{b}=1$

General form $Ax+By+C=0$ where A, B and C are integers. While your algebra teacher might have required your changing the equation $y-1=2(x-5)$ to general form $2x-y-9=0$, you will find that on the AP calculus test, it is sufficient to leave equations for a lines in point-slope form and it is recommended not to waste time changing it unless you are specifically told to do so

Parallel lines Two distinct lines are parallel if they have the same slope: $m_1=m_2$

Normal lines: Two lines are normal (perpendicular) if their slopes are negative reciprocals: $m_1 \; m_2=-1$

Horizontal lines have slope zero. **Vertical lines** have no slope (slope is undefined).

1 Find the equation of the line in slope-intercept form, with the given slope, passing through the given point.

a. $m=-4, (1,2)$

$$\boxed{y-2=-4(x-1)\Rightarrow y=-4x+6}$$

b $m=\frac{2}{3}, (-5,1)$

$$\boxed{y-1=\frac{2}{3}(x-5)\Rightarrow y=\frac{2x}{3}-\frac{7}{3}}$$

c. $m=0, \left(-\frac{1}{2},\frac{3}{4}\right)$

$$\boxed{y=-\frac{3}{4}}$$

2. Find the equation of the line in slope-intercept form, passing through the following points.

a. $(4,5)$ and $(-2,-1)$

$$\boxed{\begin{array}{l} m=\frac{5+1}{4+2}=1 \\ y-5=x-4\Rightarrow y=x+1 \end{array}}$$

b. $(0,-3)$ and $(-5,3)$

$$\boxed{\begin{array}{l} m=\frac{3+3}{-5-0}=\frac{-6}{5} \\ y+3=\frac{-6}{5}x\Rightarrow y=\frac{-6}{5}x-3 \end{array}}$$

c. $\left(\frac{3}{4},-1\right)$ and $\left(1,\frac{1}{2}\right)$

$$\boxed{\begin{array}{l} m=\left(\frac{\frac{1}{2}+1}{1-\frac{3}{4}}\right)\left(\frac{4}{4}\right)=\frac{2+4}{4-3}=6 \\ y-\frac{1}{2}=6(x-1)\Rightarrow y=6x-\frac{11}{2} \end{array}}$$

3 Write equations of the line through the given point a) parallel and b) normal to the given line.

a. $(4,7)$, $4x-2y=1$

$$\boxed{\begin{array}{l} y=2x-\frac{1}{2}\Rightarrow m=2 \\ \text{a) } y-7=2(x-4) \qquad \text{b) } y-7=\frac{-1}{2}(x-4) \end{array}}$$

b. $\left(\frac{2}{3},1\right)$, $x+5y=2$

$$\boxed{\begin{array}{l} y=\frac{-1}{5}x+2\Rightarrow m=\frac{-1}{5} \\ \text{a) } y-1=\frac{-1}{5}\left(x-\frac{2}{3}\right) \qquad \text{b) } y-1=5\left(x-\frac{2}{3}\right) \end{array}}$$

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

G. Linear Functions - Assignment

1 Find the equation of the line in slope-intercept form, with the given slope, passing through the given point.

a. $m=-7, (-3,-7)$ b. $m=\frac{-1}{2}, (2,-8)$ c. $m=\frac{2}{3}, \left(-6,\frac{1}{3}\right)$

2 Find the equation of the line in slope-intercept form, passing through the following points.

a. $(-3,6)$ and $(-1,2)$ b $(-7,1)$ and $(3,-4)$ c. $\left(-2,\frac{2}{3}\right)$ and $\left(\frac{1}{2},1\right)$

3. Write equations of the line through the given point a) parallel and b) normal to the given line.

a. $(5,-3)$, $x+y=4$ b. $(-6,2)$, $5x+2y=7$ c. $(-3,-4)$, $y=-2$

4. Find an equation of the line containing $(4,-2)$ and parallel to the line containing $(-1,4)$ and $(2,3)$ Put your answer in general form.

5. Find k if the lines $3x-5y=9$ and $2x+ky=11$ are a) parallel and b) perpendicular.

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

H. Solving Quadratic Equations

Solving quadratics in the form of $ax^2+bx+c=0$ usually show up on the AP exam in the form of expressions that can easily be factored. But occasionally, you will be required to use the quadratic formula. When you have a quadratic equation, factor it, set each factor equal to zero and solve. If the quadratic equation doesn't factor or if factoring is too time-consuming, use the quadratic formula.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The **discriminant** b^2-4ac will tell you how many solutions the quadratic has:

$$b^2-4ac\begin{cases}>0, \text{ 2 real solutions (if a perfect square, the solutions are rational)}\\ =0, \text{ 1 real solution}\\ <0, \text{ 0 real solutions (or 2 imaginary solutions, but AP calculus does not deal with imaginaries)}\end{cases}$$

1 Solve for x

a. $x^2+3x+2=0$

$$\boxed{\begin{array}{l}(x+2)(x+1)=0\\ x=-2, x=-1\end{array}}$$

b $x^2-10x+25=0$

$$\boxed{\begin{array}{l}(x-5)^2=0\\ x=5\end{array}}$$

c. $x^2-64=0$

$$\boxed{\begin{array}{l}(x-8)(x+8)=0\\ x=8, x=-8\end{array}}$$

d. $2x^2+9x=18$

$$\boxed{\begin{array}{l}(2x-3)(x+6)=0\\ x=\frac{3}{2}, x=-6\end{array}}$$

e. $12x^2+23x=-10$

$$\boxed{\begin{array}{l}(4x+5)(3x+2)=0\\ x=-\frac{5}{4}, x=-\frac{2}{3}\end{array}}$$

f. $48x-64x^2=9$

$$\boxed{\begin{array}{l}(8x-3)^2=0\\ x=\frac{3}{8}\end{array}}$$

g. $x^2+5x=2$

$$\boxed{\begin{array}{l}x=\frac{-5\pm\sqrt{25+8}}{2}\\ x=\frac{-5\pm\sqrt{33}}{2}\end{array}}$$

h. $8x-3x^2=2$

$$\boxed{\begin{array}{l}x=\frac{8\pm\sqrt{64-24}}{6}\\ x=\frac{8\pm2\sqrt{10}}{6}=\frac{4\pm\sqrt{10}}{3}\end{array}}$$

i. $6x^2+5x+3=0$

$$\boxed{\begin{array}{l}x=\frac{-5\pm\sqrt{25-72}}{12}=\frac{-5\pm\sqrt{-47}}{12}\\ \text{No real solutions}\end{array}}$$

j $x^3-3x^2+3x-9=0$

$$\boxed{\begin{array}{l}x^2(x-3)-3(x-3)=0\\ (x-3)(x^2-3)=0\\ x=3,\ x=\pm\sqrt{3}\end{array}}$$

k. $\frac{x}{3}-\frac{5}{2}=\frac{-3}{x}$

$$\boxed{\begin{array}{l}6x\left(\frac{x}{3}-\frac{5}{2}=\frac{-3}{x}\right)\\ 2x^2-15x+18=0\\ (2x-3)(x-6)=0\\ x=\frac{3}{2},\ x=6\end{array}}$$

l. $x^4-7x^2-8=0$

$$\boxed{\begin{array}{l}(x^2-8)(x^2+1)=0\\ x=\pm\sqrt{8}=\pm2\sqrt{2}\end{array}}$$

2 If $y=5x^2-3x+k$, for what values of k will the quadratic have two real solutions?

$$\boxed{(-3)^2-4(5)k>0\Rightarrow 9-20k>0\Rightarrow k<\frac{9}{20}}$$

ODD PROBLEMS ONLY

Name: ______________________ Class: ___________ Date: ___________ Score: ___________

H. Solving Quadratic Equations Assignment

1 Solve for x.

a. $x^2+7x-18=0$

b. $x^2+x+\frac{1}{4}=0$

c. $2x^2-72=0$

d. $12x^2-5x=2$

e. $20x^2-56x+15=0$

f. $81x^2+72x+16=0$

g. $x^2+10x=7$

h. $3x-4x^2=-5$

i. $7x^2-7x+2=0$

j. $x+\frac{1}{x}=\frac{17}{4}$

k. $x^3-5x^2+5x-25=0$

l. $2x^4-15x^3+18x^2=0$

2 If $y=x^2+kx-k$, for what values of k will the quadratic have two real solutions?

3 Find the domain of $y=\frac{2x-1}{6x^2-5x-6}$

ODD PROBLEMS ONLY

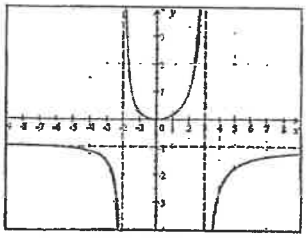
Name: ____________________ Class: __________ Date: __________ Score: __________

I. Asymptotes

Rational functions in the form of $y = \frac{p(x)}{q(x)}$ possibly have vertical asymptotes, lines that the graph of the curve approach but never cross. To find the **vertical asymptotes**, factor out any common factors of numerator and denominator, reduce if possible, and then set the denominator equal to zero and solve.

Horizontal asymptotes are lines that the graph of the function approaches when x gets very large or very small. While you learn how to find these in calculus, a rule of thumb is that if the highest power of x is in the denominator, the horizontal asymptote is the line $y = 0$ If the highest power of x is both in numerator and denominator, the horizontal asymptote will be the line $y = \frac{\text{highest degree coefficient in numerator}}{\text{highest degree coefficient in denominator}}$ If the highest power of x is in the numerator, there is no horizontal asymptote, but a slant asymptote which is not used in calculus.

1) Find any vertical and horizontal asymptotes for the graph of $y = \frac{-x^2}{x^2 - x - 6}$

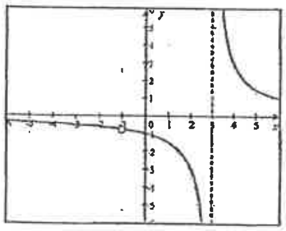


$$y = \frac{-x^2}{x^2 - x - 6} = \frac{-x^2}{(x-3)(x+2)}$$

Vertical asymptotes. $x - 3 = 0 \Rightarrow x = 3$ and $x + 2 = 0 \Rightarrow x = -2$
Horizontal asymptotes: Since the highest power of x is 2 in both numerator and denominator, there is a horizontal asymptote at $y = -1$

This is confirmed by the graph to the right Note that the curve actually crosses its horizontal asymptote on the left side of the graph.

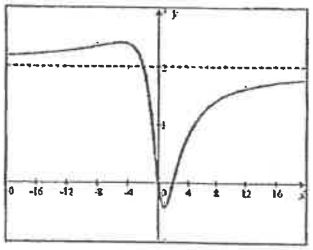
2) Find any vertical and horizontal asymptotes for the graph of $y = \frac{3x+3}{x^2 - 2x - 3}$



$$y = \frac{3x+3}{x^2 - 2x - 3} = \frac{3(x+1)}{(x-3)(x+1)} = \frac{3}{x-3}$$

Vertical asymptotes: $x - 3 = 0 \Rightarrow x = 3$ Note that since the $(x+1)$ cancels, there is no vertical asymptote at $x = 1$, but a hole (sometimes called a removable discontinuity) in the graph.
Horizontal asymptotes. Since there the highest power of x is in the denominator, there is a horizontal asymptote at $y = 0$ (the x-axis). This is confirmed by the graph to the right.

3) Find any vertical and horizontal asymptotes for the graph of $y = \frac{2x^2 - 4x}{x^2 + 4}$



$$y = \frac{2x^2 - 4x}{x^2 + 4} = \frac{2x(x-2)}{x^2 + 4}$$

Vertical asymptotes. None. The denominator doesn't factor and setting it equal to zero has no solutions.
Horizontal asymptotes: Since the highest power of x is 2 in both numerator and denominator, there is a horizontal asymptote at $y = 2$. This is confirmed by the graph to the right.

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

I. Asymptotes - Assignment

- Find any vertical and horizontal asymptotes and if present, the location of holes, for the graph of

1. $y=\frac{x-1}{x+5}$

2. $y=\frac{8}{x^2}$

3 $y=\frac{2x+16}{x+8}$

4. $y=\frac{2x^2+6x}{x^2+5x+6}$

5. $y=\frac{x}{x^2-25}$

6. $y=\frac{x^2-5}{2x^2-12}$

7 $y=\frac{4+3x-x^2}{3x^2}$

8. $y=\frac{5x+1}{x^2-x-1}$

9. $y=\frac{1-x-5x^2}{x^2+x+1}$

10. $y=\frac{x^3}{x^2+4}$

11. $y=\frac{x^3+4x}{x^3-2x^2+4x-8}$

12. $y=\frac{10x+20}{x^3-2x^2-4x+8}$

13. $y=\frac{1}{x}-\frac{x}{x+2}$ (hint: express with a common denominator)

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

J. Negative and Fractional Exponents

In calculus, you will be required to perform algebraic manipulations with **negative exponents** as well as **fractional exponents**. You should know the definition of a negative exponent: $x^{-n} = \frac{1}{x^n}, x \neq 0$ Note that negative powers do not make expressions negative; they create fractions. Typically expressions in multiple-choice answers are written with positive exponents and students are required to eliminate negative exponents. Fractional exponents create roots. The definition of $x^{1/2} = \sqrt{x}$ and $x^{a/b} = \sqrt[b]{x^a} = \left(\sqrt[b]{x}\right)^a$

As a reminder when we multiply, we add exponents. $\left(x^a\right)\left(x^b\right) = x^{a+b}$

When we divide, we subtract exponents: $\frac{x^a}{x^b} = x^{a-b}, x \neq 0$

When we raise powers, we multiply exponents: $\left(x^a\right)^b = x^{ab}$

In your algebra course, leaving an answer with a radical in the denominator was probably not allowed. You had to rationalize the denominator $\frac{1}{\sqrt{x}}$ changed to $\left(\frac{1}{\sqrt{x}}\right)\left(\frac{\sqrt{x}}{\sqrt{x}}\right) = \frac{\sqrt{x}}{x}$ In calculus, you will find that it is not necessary to rationalize and it is recommended that you not take the time to do so.

- Simplify and write with positive exponents. Note: # 12 involves complex fractions, covered in section K.

1 $-8x^{-2}$

$$\boxed{\frac{-8}{x^2}}$$

2. $\left(-5x^3\right)^{-2}$

$$\boxed{(-5)^{-2} x^{-6} = \frac{1}{(-5)^2 x^6} = \frac{1}{25x^6}}$$

3 $\left(\frac{-3}{x^4}\right)^{-2}$

$$\boxed{\frac{(-3)^{-2}}{\left(x^4\right)^{-2}} = \frac{1}{(-3)^2 x^{-8}} = \frac{x^8}{9}}$$

4 $\left(36x^{10}\right)^{1/2}$

$$\boxed{6x^5}$$

5 $\left(27x^3\right)^{-2/3}$

$$\boxed{\frac{1}{\left(27x^3\right)^{2/3}} = \frac{1}{9x^2}}$$

6. $\left(16x^{-2}\right)^{3/4}$

$$\boxed{16^{3/4} x^{-4/3} = \frac{8}{x^{4/3}}}$$

7 $\left(x^{1/2} - x\right)^{-2}$

$$\boxed{\frac{1}{\left(x^{1/2} - x\right)^2} = \frac{1}{x - 2x^{3/2} + x^2}}$$

8. $\left(4x^2 - 12x + 9\right)^{-1/2}$

$$\boxed{\frac{1}{\left[(2x-3)^2\right]^{1/2}} = \frac{1}{2x-3}}$$

9 $\left(x^{1/3}\right)\left(\frac{1}{2}x^{-1/2}\right) + \left(x^{1/2} + 1\right)\left(\frac{1}{3}x^{-1/3}\right)$

$$\boxed{\frac{x^{1/3}}{2x^{1/2}} + \frac{x^{1/2}+1}{3x^{1/3}} = \frac{1}{2x^{1/6}} + \frac{x^{1/2}+1}{3x^{1/3}}}$$

10 $\frac{-2}{3}(8x)^{-5/3}(8)$

$$\boxed{\frac{-16}{3(8x)^{5/3}} = \frac{-16}{3(32)x^{5/3}} = -\frac{1}{6x^{5/3}}}$$

11 $\frac{(x+4)^{1/2}}{(x-4)^{-1/2}}$

$$\boxed{(x+4)^{1/2}(x-4)^{1/2} = \left(x^2 - 16\right)^{1/2}}$$

12. $\left(x^{-1} + y^{-1}\right)^{-1}$

$$\boxed{\left(\frac{1}{\frac{1}{x} + \frac{1}{y}}\right)\left(\frac{xy}{xy}\right) = \frac{xy}{y+x}}$$

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

J. Negative and Fractional Exponents - Assignment

Simplify and write with positive exponents.

1. -12^2x^{-5}

2. $\left(-12x^5\right)^{-2}$

3. $\left(4x^{-1}\right)^{-1}$

4. $\left(\frac{-4}{x^4}\right)^{-3}$

5. $\left(\frac{5x^3}{y^2}\right)^{-3}$

6. $\left(x^3-1\right)^{-2}$

7 $\left(121x^8\right)^{1/2}$

8. $\left(8x^2\right)^{-4/3}$

9. $\left(-32x^{-5}\right)^{-3/5}$

10. $(x+y)^{-2}$

11. $\left(x^3+3x^2+3x+1\right)^{-2/3}$

12. $x\left(x^{1/2}-x\right)^{-2}$

13. $\frac{1}{4}\left(16x^2\right)^{-3/4}(32x)$

14. $\frac{\left(x^2-1\right)^{-1/2}}{\left(x^2+1\right)^{1/2}}$

15. $\left(x^{-2}+2^{-2}\right)^{-1}$

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

K. Eliminating Complex Fractions

Calculus frequently uses **complex fractions,** which are fractions within fractions. Answers are never left with complex fractions and they must be eliminated. There are two methods to eliminate complex fractions.

When the problem is in the form of $\frac{\frac{a}{b}}{\frac{c}{d}}$, we can "flip the denominator" and write it as $\frac{a}{b}\frac{d}{c}=\frac{ad}{bc}$

However, this does not work when the numerator and denominator are not single fractions. The best way is to eliminate the complex fractions in all cases is to find the LCD (lowest common denominator) of all the fractions in the complex fraction. Multiply all terms by this LCD and you are left with a fraction that is magically no longer complex. **Important:** Note that $\frac{x^{-1}}{y^{-1}}$ can be written as $\frac{y}{x}$ but $\frac{1+x^{-1}}{y^{-1}}$ must be written as $\frac{1+\frac{1}{x}}{\frac{1}{y}}$

- Eliminate the complex fractions.

1 $\dfrac{\frac{2}{3}}{\frac{5}{6}}$

$$\boxed{\left(\frac{\frac{2}{3}}{\frac{5}{6}}\right)\left(\frac{6}{6}\right)=\frac{4}{5}}$$

2. $\dfrac{1+\frac{2}{3}}{1+\frac{5}{6}}$

$$\boxed{\left(\frac{1+\frac{2}{3}}{1+\frac{5}{6}}\right)\left(\frac{6}{6}\right)=\frac{6+4}{6+5}=\frac{10}{11}}$$

3 $\dfrac{\frac{3}{4}+\frac{5}{3}}{2-\frac{1}{6}}$

$$\boxed{\left(\frac{\frac{3}{4}+\frac{5}{3}}{2-\frac{1}{6}}\right)\left(\frac{12}{12}\right)=\frac{9+20}{24-2}=\frac{29}{22}}$$

4 $\dfrac{1+\frac{1}{2}x^{-1}}{1+\frac{1}{3}x^{-1}}$

$$\boxed{\left(\frac{1+\frac{1}{2x}}{1+\frac{1}{3x}}\right)\left(\frac{6x}{6x}\right)=\frac{6x+3}{6x+2}}$$

5 $\dfrac{x-\frac{1}{2x}}{x^2+\frac{1}{4x^2}}$

$$\boxed{\left(\frac{x-\frac{1}{2x}}{x^2+\frac{1}{4x^2}}\right)\left(\frac{4x^2}{4x^2}\right)=\frac{4x^3-2x}{4x^4+1}}$$

6. $\dfrac{\frac{2}{5}x^{5/3}}{\frac{5}{3}}$

$$\boxed{\left(\frac{\frac{2}{5}x^{5/3}}{\frac{5}{3}}\right)\left(\frac{15}{15}\right)=\frac{6x^{5/3}}{25}}$$

7 $\dfrac{x^{-3}+x}{x^{-2}+1}$

$$\boxed{\left(\frac{\frac{1}{x^3}+x}{\frac{1}{x^2}+1}\right)\left(\frac{x^3}{x^3}\right)=\frac{1+x^4}{x+x^3}}$$

8. $\dfrac{\frac{1}{2}(2x+5)^{-2/3}}{\frac{-2}{3}}$

$$\boxed{\left(\frac{\frac{1}{2}}{\frac{-2}{3}(2x+5)^{2/3}}\right)\frac{6}{6}=\frac{-3}{4(2x+5)^{2/3}}}$$

9 $\dfrac{(x-1)^{1/2}-\frac{x(x-1)^{-1/2}}{2}}{x-1}$

$$\boxed{\left(\frac{(x-1)^{1/2}-\frac{x}{2(x-1)^{1/2}}}{x-1}\right)\left[\frac{2(x-1)^{1/2}}{2(x-1)^{1/2}}\right] \quad \frac{2(x-1)-x}{2(x-1)^{3/2}}=\frac{x-2}{2(x-1)^{3/2}}}$$

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

K. Eliminating Complex Fractions - Assignment

- Eliminate the complex fractions.

1 $\dfrac{\frac{5}{8}}{\frac{-2}{3}}$

2. $\dfrac{4-\frac{2}{9}}{3+\frac{4}{3}}$

3. $\dfrac{2+\frac{7}{2}+\frac{3}{5}}{5-\frac{3}{4}}$

4. $\dfrac{x-\frac{1}{x}}{x+\frac{1}{x}}$

5. $\dfrac{1+x^{-1}}{1-x^{-2}}$

6. $\dfrac{x^{-1}+y^{-1}}{x+y}$

7. $\dfrac{x^{-2}+x^{-1}+1}{x^{-2}-x}$

8. $\dfrac{\frac{1}{3}(3x-4)^{-3/4}}{\frac{-3}{4}}$

9 $\dfrac{2x(2x-1)^{1/2}-2x^{2}(2x-1)^{-1/2}}{(2x-1)}$

ODD PROBLEMS ONLY

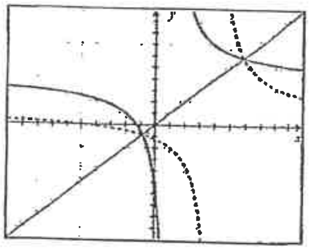
Name: ____________________ Class: __________ Date: __________ Score: __________

L. Inverses

No topic in math confuses students more than inverses. If a function is a rule that maps x to y, an **inverse** is a rule that brings y back to the original x. If a point (x,y) is a point on a function f, then the point (y,x) is on the inverse function f^{-1} Students mistakenly believe that since $x^{-1}=\frac{1}{x}$, then $f^{-1}=\frac{1}{f}$ This is decidedly incorrect. If a function is given in equation form, to find the inverse, replace all occurrences of x with y and all occurrences of y with x If possible, then solve for y Using the "horizontal-line test" on the original function f will quickly determine whether or not f^{-1} is also a function. By definition, $f\left(f^{-1}(x)\right)=x$ The domain of f^{-1} is the range of f and the range of f^{-1} is the domain of f

1 Find the inverse to $y=\frac{4x+5}{x-1}$ and show graphically that its inverse is a function.

Inverse: $x=\frac{4y+5}{y-1}\Rightarrow xy-x=4y+5\Rightarrow y=\frac{x+5}{x-4}$



Note that the function is drawn in bold and the inverse as dashed. The function and its inverse is symmetrical to the line $y=x$. The inverse is a function for two reasons. a) it passes the vertical line text or b) the function passes the horizontal line test.

2. Find the inverse to the following functions and show graphically that its inverse is a function.

a. $y=4x-3$

Inverse: $x=4y-3$
$y=\frac{x+3}{4}$ (function)

b. $y=x^2+1$

Inverse: $x=y^2+1$
$y=\pm\sqrt{x-1}$ (not a function)

c. $y=x^2+4x+4$

Inverse: $x=y^2+4y+4$
$x=(y+2)^2\Rightarrow\pm\sqrt{x}=y+2$
$y=-2\pm\sqrt{x}$ (not a function)

3. Find the inverse to the following functions and show that $f\left(f^{-1}(x)\right)=x$

a. $f(x)=7x+4$

Inverse: $x=7y+4$
$y=f^{-1}(x)=\frac{x-4}{7}$
$f\left(\frac{x-4}{7}\right)=7\left(\frac{x-4}{7}\right)+4=x$

b. $f(x)=\frac{1}{x-1}$

Inverse: $x=\frac{1}{y-1}\Rightarrow xy-x=1$
$y=f^{-1}(x)=\frac{x+1}{x}$
$f\left(\frac{x+1}{x}\right)=\left(\frac{1}{\frac{x+1}{x}-1}\right)\left(\frac{x}{x}\right)$
$=\frac{x}{x+1-x}=x$

c. $f(x)=x^3-1$

Inverse: $x=y^3-1$
$y=f^{-1}(x)=\sqrt[3]{x+1}$
$f\left(\sqrt[3]{x+1}\right)=\left(\sqrt[3]{x+1}\right)^3-1=x$

4 Without finding the inverse, find the domain and range of the inverse to $f(x)=\sqrt{x-2}+3$

Function. Domain: $[-2,\infty)$, Range. $[3,\infty)$ Inverse: Domain. $[3,\infty)$, Range: $[-2,\infty)$

RU Ready?

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

L. Inverses – Assignment

1 Find the inverse to the following functions and show graphically that its inverse is a function.

a. $2x-6y=1$

b. $y=ax+b$

c. $y=9-x^2$

d. $y=\sqrt{1-x^3}$

e. $y=\frac{9}{x}$

f. $y=\frac{2x+1}{3-2x}$

2. Find the inverse to the following functions and show that $f\left(f^{-1}(x)\right)=x$

a. $f(x)=\frac{1}{2}x-\frac{4}{5}$

b $f(x)=x^2-4$

c. $f(x)=\frac{x^2}{x^2+1}$

3. Without finding the inverse, find the domain and range of the inverse to $f(x)=\frac{\sqrt{x+1}}{x^2}$

RU Ready?

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

M. Adding Fractions and Solving Fractional Equations

There are two major problem types with fractions. Adding/subtracting fractions and solving fractional equations. Algebra has taught you that in order to add fractions, you need to find an LCD and *multiply each fraction by one* in such a way that you obtain the LCD in each fraction. However, when you solve fractional equations (equations that involve fractions), you still find the LCD but you *multiply every term by the LCD* When you do that, all the fractions disappear, leaving you with an equation that is hopefully solvable. Answers should be checked in the original equation.

1 a. Combine: $\frac{x}{3}-\frac{x}{4}$

LCD· 12 $\frac{x}{3}\left(\frac{4}{4}\right)-\frac{x}{4}\left(\frac{3}{3}\right)$

$$\frac{4x-3x}{12}=\frac{x}{12}$$

b. Solve: $\frac{x}{3}-\frac{x}{4}=12$

$$12\left(\frac{x}{3}\right)-12\left(\frac{x}{4}\right)=12(12)$$

$$4x-3x=144\Rightarrow x=144$$

$$x=144\quad \frac{144}{3}-\frac{144}{4}=48-36=12$$

2. a. Combine $x+\frac{6}{x}$

LCD· x $\quad x\left(\frac{x}{x}\right)+\frac{6}{x}$

$$\frac{x^2+6}{x}$$

b. Solve. $x+\frac{6}{x}=5$

$$x(x)+x\left(\frac{6}{x}\right)=5x$$

$$x^2+6=5x\Rightarrow x^2-5x+6=0$$

$$(x-2)(x-3)=0\Rightarrow x=2,\ x=3$$

$$x=2\quad 2+\frac{6}{2}=2+3=5\quad x=3\quad 3+\frac{6}{3}=3+2=5$$

3 a. Combine: $\frac{12}{x+2}-\frac{4}{x}$

LCD· $x(x+2)$ $\quad\left(\frac{12}{x+2}\right)\left(\frac{x}{x}\right)-\frac{4}{x}\left(\frac{x+2}{x+2}\right)$

$$\frac{12x-4x-8}{x(x+2)}$$

$$\frac{8x-8}{x(x+2)}$$

b. Solve $\frac{12}{x+2}-\frac{4}{x}=1$

$$\frac{12}{x+2}(x)(x+2)-\frac{4}{x}(x)(x+2)=1(x)(x+2)$$

$$12x-4x-8=x^2+2x\Rightarrow x^2-6x+8=0$$

$$(x-2)(x-4)=0\Rightarrow x=2,4$$

$$x=2\quad \frac{12}{4}-\frac{4}{2}=3-2=1\qquad x=4\quad \frac{12}{6}-\frac{4}{4}=2-1=1$$

4 a. $\frac{x}{2x-6}-\frac{3}{x^2-6x+9}$

LCD· $2(x-3)^2$

$$\frac{x}{2(x-3)}\left(\frac{x-3}{x-3}\right)-\frac{3}{(x-3)^2}\left(\frac{2}{2}\right)$$

$$\frac{x^2-3x-6}{2(x-3)^2}$$

b. Solve $\frac{x}{2x-6}-\frac{3}{x^2-6x+9}=\frac{x-2}{3x-9}$

$$\left[\frac{x}{2(x-3)}-\frac{3}{(x-3)^2}=\frac{x-2}{3(x-3)}\right]6(x-3)^2$$

$$3x(x-3)-18=2(x-3)(x-2)$$

$$3x^2-9x-18=2x^2-10x+12$$

$$x^2+x-30=0\Rightarrow(x+6)(x-5)=0\Rightarrow x=-6,5$$

$$x=-6\quad \frac{-6}{-18}-\frac{3}{81}=\frac{-8}{-27}\qquad x=5\quad \frac{5}{4}-\frac{3}{4}=\frac{3}{6}$$

RU Ready?

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

M. Adding Fractions and Solving Fractional Equations - Assignment

1 a. Combine. $\frac{2}{3}-\frac{1}{x}$

b Solve: $\frac{2}{3}-\frac{1}{x}=\frac{5}{6}$

2. a. Combine: $\frac{1}{x-3}+\frac{1}{x+3}$

b Solve: $\frac{1}{x-3}+\frac{1}{x+3}=\frac{10}{x^2-9}$

3 a. Combine: $\frac{5}{2x}-\frac{5}{3x+15}$

b. Solve: $\frac{5}{2x}-\frac{5}{3(x+5)}=\frac{5}{x}$

4 a. Combine: $\frac{2x-1}{x-1}-\frac{3x}{2x+1}$

b Solve: $\frac{2x-1}{x-1}-\frac{3x}{2x+1}=\frac{x^2+11}{2x^2-x-1}$

RU Ready?

ODD PROBLEMS ONLY

Name: ______________________ Class: ___________ Date: ___________ Score: ___________

N. Solving Absolute Value Equations

Absolute value equations crop up in calculus, especially in BC calculus. The definition of the absolute value function is a piecewise function. $f(x)=|x|=\begin{cases} x & \text{if } x\ge 0 \\ -x & \text{if } x<0 \end{cases}$ So, to solve an absolute value equation, split the absolute value equation into two equations, one with a positive parentheses and the other with a negative parentheses and solve each equation. It is possible that this procedure can lead to incorrect solutions so solutions must be checked.

- Solve the following equations.

1 $|x-1|=3$

$x-1=3$	$-(x-1)=3$
$x=4$	$-x+1=3$
	$x=-2$

2. $|3x+2|=9$

	$-(3x+2)=9$
$3x+2=9$	$-3x-2=9$
$3x=7$	$3x=-11$
$x=\frac{7}{3}$	$x=\frac{-11}{3}$

3. $|2x-1|-x=5$

$2x-1-x=5$	$-(2x-1)-x=5$
$x=6$	$-3x=4$
	$x=\frac{-4}{3}$

4 $|x+5|+5=0$

$x+5+5=0$	$-(x+5)+5=0$
$x=-10$	$-x-5+5=0$
	$x=0$

Both answers are invalid. It is impossible to add 5 to an absolute value and get 0

5 $|x^2-x|=2$

$(x^2-x)=2$	$-(x^2-x)=2$
$x^2-x-2=0$	$-x^2+x=2$
$(x-2)(x+1)=0$	$0=x^2+x+2$
$x=2, x=-1$	No real solution

Both solutions check

6. $|x-10|=x^2-10x$

$x-10=x^2-10x$	$-(x-10)=x^2-10x$
$x^2-11x+10=0$	$-x+10=x^2-10x$
$(x-1)(x-10)=0$	$x^2-9x-10=0$
$x=1, x=10$	$(x-10)(x+1)=0$
	$x=10, x=-1$

Of the three solutions, only $x=-1$ and $x=10$ are valid.

7 $|x|+|2x-2|=8$

$x+2x-2=8$	$-x+2x-2=8$	$x-(2x-2)=8$	$-x-(2x-2)=8$
$3x=10$	$x=10$	$-x=6$	$-3x=6$
$x=\frac{10}{3}$		$x=-6$	$x=-2$

Of the four solutions, only $x=\frac{10}{3}$ and $x=-2$ are valid

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

N. Solving Absolute Value Equations - Assignment

- Solve the following equations.

1. $4|x+8|=20$

2. $|1-7x|=13$

3. $|8+2x|+2x=40$

4. $|4x-5|+5x+2=0$

5. $|x^2-2x-1|=7$

6 $|12-x|=x^2-12x$

7. $|x|+|4x-4|+x=14$

RU Ready?

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

O. Solving Inequalities

You may think that solving inequalities are just a matter of replacing the equal sign with an inequality sign. In reality, they can be more difficult and are fraught with dangers. And in calculus, inequalities show up more frequently than solving equations. Solving inequalities are a simple matter if they are based on linear equations. They are solved exactly like linear equations, remembering that if you multiply or divide both sides by a negative number, the direction of the inequality sign must be reversed.

However, if the inequality is more complex than a linear function, it is advised to bring all terms to one side. Pretend for a moment it is an equation and solve. Then create a number line which determines whether the transformed inequality is positive or negative in the intervals created on the number line and choose the correct intervals according to the inequality, paying attention to whether the zeroes are included or not.

If the inequality involves an absolute value, create two equations, replacing the absolute value with a positive parentheses and a negative parentheses and the inequality sign with an equal sign. Solve each, placing each solution on your number line. Then determine which intervals satisfy the original inequality

If the inequality involves a rational function, set both numerator and denominator equal to zero, which will give you the values you need for your number line. Determine whether the inequality is positive or negative in the intervals created on the number line and choose the correct intervals according to the inequality, paying attention to whether the endpoints are included or not.

- Solve the following inequalities.

1 $2x-8\le 6x+2$

$-10\le 4x \qquad -4x\le 10$

$\frac{-5}{2}\le x \quad \text{or} \quad x\ge\frac{-5}{2}$

2. $1-\frac{3x}{2}>x-5$

$2-3x>2x-10$

$12>5x\Rightarrow x<\frac{12}{5}$

3 $-5\le 6x-1<11$

$-6\le 6x\le 12$

$-1\le x\le 2$

4. $|2x-1|\le x+4$

$|2x-1|-x-4\le 0$

$2x-1-x-4=0 \qquad -2x+1-x-4=0$

$x=5 \qquad x=-1$

$+++++++0-----------0++++++++$

$1 \qquad 5$

So $-1\le x\le 5$ or $[-1,5]$

5 $x^2-3x>18$

$x^2-3x-18>0\Rightarrow(x+3)(x-6)>0$

For $(x+3)(x-6)=0$, $x=-3, x=6$

$++++++0--------0+++++++$

$\cdot 3 \qquad 6$

So $x<-3$ or $x>6$ or $(-\infty,-3)\cup(6,\infty)$

6. $\frac{2x-7}{x-5}\le 1$

$\frac{2x-7}{x-5}-1=0\Rightarrow\frac{2x-7}{x-5}-\frac{x-5}{x-5}<0\Rightarrow\frac{x-2}{x-5}<0$

$++++++0---------\infty+++++++$

$2 \qquad 5$

So $2\le x<5$ or $[2,5)$

7 Find the domain of $\sqrt{32-2x^2}$

$2(4+x)(4-x)\ge 0$

$-------0+++++++0--------$

$-4 \qquad 4$

So $-4\le x\le -4$ or $[-4,4]$

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

O. Solving Inequalities - Assignment

- Solve the following inequalities.

1. $5(x-3) \leq 8(x+5)$

2. $4-\frac{5x}{3} > -\left(2x+\frac{1}{2}\right)$

3 $\frac{3}{4} > x+1 > \frac{1}{2}$

4. $x+7 \geq |5-3x|$

5. $(x+2)^2 < 25$

6. $x^3 < 4x^2$

7. $\frac{5}{x-6} \geq \frac{1}{x+2}$

8. Find the domain of $\sqrt{\frac{x^2-x-6}{x-4}}$

RU Ready?

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

P. Exponential Functions and Logarithms

Calculus spends a great deal of time on exponential functions in the form of b^x Don't expect that when you start working with them in calculus, your teacher will review them. So learn them now! Students must know that the definition of a **logarithm** is based on exponential equations. If $y = b^x$ then $x = \log_b y$ So when you are trying to find the value of $\log_2 32$, state that $\log_2 32 = x$ and $2^x = 32$ and therefore $x = 5$

If the base of a log statement is not specified, it is defined to be 10 When we asked for log 100, we are solving the equation. $10^x = 100$ and $x = 2$. The function $y = \log x$ has domain $(0,\infty)$ and range $(-\infty,\infty)$ In calculus, we primarily use logs with base *e*, which are called natural logs (ln). So finding ln 5 is the same as solving the equation $e^x = 5$ Students should know that the value of $e = 2.71828$.

There are three rules that students must keep in mind that will simplify problems involving logs and natural logs. These rules work with logs of any base including natural logs.

i. $\log a + \log b = \log(a\ \ b)$ ii. $\log a - \log b = \log\left(\frac{a}{b}\right)$ iii. $\log a^b = b \log a$

1 Find a. $\log_4 8$

$\log_4 8 = x$

$4^x = 8 \Rightarrow 2^{2x} = 2^3$

$x = \frac{3}{2}$

b. $\ln\sqrt{e}$

$\ln\sqrt{e} = x$

$e^x = e^{1/2}$

$x = \frac{1}{2}$

c. $10^{\log 4}$

$\log 4 = x$

$10^x = 4$ so $10^{\log 4} = 4$

10 to a power and log are inverses

d. $\log 2 + \log 50$

$\log(2\ \ 50) = \log 100$

2

e. $\log_4 192 - \log_4 3$

$\log_4\left(\frac{192}{3}\right)$

$\log_4 64 = 3$

f. $\ln\sqrt[5]{e^3}$

$\ln e^{3/5} = \frac{3}{5}\ln e = \frac{3}{5}$

2. Solve a. $\log_9\left(x^2 - x + 3\right) = \frac{1}{2}$

$x^2 - x + 3 = 9^{1/2}$

$x(x-1) = 0$

$x = 0, x = 1$

b. $\log_{36} x + \log_{36}(x-1) = \frac{1}{2}$

$\log_{36} x(x-1) = \frac{1}{2}$

$x(x-1) = 36^{1/2} = 6$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

Only $x = 3$ is in the domain

c. $\ln x - \ln(x-1) = 1$

$\ln\left(\frac{x}{x-1}\right) = 1$

$\frac{x}{x-1} = e \Rightarrow x = ex - e$

$x = \frac{e}{e-1}$

d. $5^x = 20$

$\log(5^x) = \log 20$

$x \log 5 = \log 20$

$x = \frac{\log 20}{\log 5}$ or $x = \frac{\ln 20}{\ln 5}$

e. $e^{-2x} = 5$

$\ln e^{-2x} = \ln 5$

$-2x = \ln 5 \Rightarrow x = \frac{-\ln 5}{2}$

f. $2^x = 3^{x-1}$

$\log(2^x) = \log(3^{x-1})$

$x \log 2 = (x-1)\log 3$

$x \log 2 = x \log 3 - \log 3 \Rightarrow x = \frac{\log 3}{\log 3 - \log 2}$

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

P. Exponential Functions and Logarithms - Assignment

1 Find a. $\log_2 \frac{1}{4}$

b $\log_8 4$

c. $\ln \frac{1}{\sqrt[3]{e^2}}$

d. $5^{\log_5 40}$

e. $e^{\ln 12}$

f. $\log_{12} 2 + \log_{12} 9 + \log_{12} 8$

g. $\log_2 \frac{2}{3} + \log_2 \frac{3}{32}$

h. $\log_{\frac{1}{3}} \frac{4}{3} - \log_{\frac{1}{3}} 12$

i. $\log_5 \left(\sqrt{3}\right)^5$

2. Solve a. $\log_5(3x-8)=2$

b. $\log_9\left(x^2-x+3\right)=\frac{1}{2}$

c. $\log(x-3)+\log 5=2$

d. $\log_2(x-1)+\log_2(x+3)=5$

e. $\log_5(x+3)-\log_5 x=2$

f. $\ln x^3 - \ln x^2 = \frac{1}{2}$

g. $3^{x-2}=18$

h. $e^{3x+1}=10$

i. $8^x = 5^{2x-1}$

RU Ready?

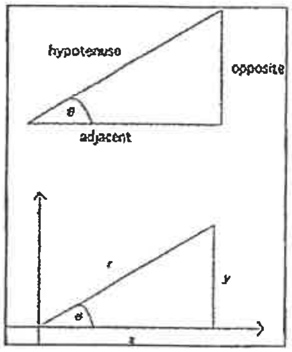
ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

Q. Right Angle Trigonometry

Trigonometry is an integral part of AP calculus. Students must know the basic trig function definitions in terms of opposite, adjacent and hypotenuse as well as the definitions if the angle is in standard position.

Given a right triangle with one of the angles named θ, and the sides of the triangle relative to θ named opposite (y), adjacent (x), and hypotenuse (r) we define the 6 trig functions to be:



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} \qquad \csc\theta = \frac{\text{hyptotenuse}}{\text{opposite}} = \frac{r}{y}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r} \qquad \sec\theta = \frac{\text{hyptotenuse}}{\text{adjacent}} = \frac{r}{x}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{x}{y}$$

The Pythagorean theorem ties these variables together: $x^2 + y^2 = r^2$ Students should recognize right triangles with integer sides. 3-4-5, 5-12-13, 8-15-17, 7-24-25 Also any multiples of these sides are also sides of a right triangle. Since r is the largest side of a right triangle, it can be shown that the range of $\sin\theta$ and $\cos\theta$ is $[-1,1]$, the range of $\csc\theta$ and $\sec\theta$ is $(-\infty,-1]\cup[1,\infty)$ and the range of $\tan\theta$ and $\cot\theta$ is $(-\infty,\infty)$.

Also vital to master is the signs of the trig functions in the four quadrants. A good way to remember this is A – S – T – C where <u>A</u>ll trig functions are positive in the 1st quadrant, <u>S</u>in is positive in the 2nd quadrant, <u>T</u>an is positive in the 3rd quadrant and <u>C</u>os is positive in the 4th quadrant.

1 Let be a point on the terminal side of θ Find the 6 trig functions of θ (Answers need not be rationalized).

a) $P(-8,6)$

$x=-8, y=6, r=10$

$\sin\theta = \frac{3}{5}$ $\quad\csc\theta = \frac{5}{3}$

$\cos\theta = -\frac{4}{5}$ $\quad\sec\theta = -\frac{5}{4}$

$\tan\theta = -\frac{3}{4}$ $\quad\cot\theta = -\frac{4}{3}$

b $P(1,3)$

$x=1,\ y=3,\ r=\sqrt{10}$

$\sin\theta = \frac{3}{\sqrt{10}}$ $\quad\csc\theta = \frac{\sqrt{10}}{3}$

$\cos\theta = \frac{1}{\sqrt{10}}$ $\quad\sec\theta = \sqrt{10}$

$\tan\theta = 3$ $\quad\cot\theta = \frac{1}{3}$

c. $P(-\sqrt{10},-\sqrt{6})$

$x=-\sqrt{10},\ y=-\sqrt{6},\ r=4$

$\sin\theta = -\frac{\sqrt{6}}{4}$ $\quad\csc\theta = -\frac{4}{\sqrt{6}}$

$\cos\theta = -\frac{\sqrt{10}}{4}$ $\quad\sec\theta = -\frac{4}{\sqrt{10}}$

$\tan\theta = \sqrt{\frac{3}{5}}$ $\quad\cot\theta = \sqrt{\frac{5}{3}}$

2. If $\cos\theta = \frac{2}{3}$, θ in quadrant IV, find $\sin\theta$ and $\tan\theta$

$x=2,\ r=3,\ y=-\sqrt{5}$

$\sin\theta = -\frac{\sqrt{5}}{3}, \tan\theta = -\frac{\sqrt{5}}{2}$

3 If $\sec\theta = \sqrt{3}$ find $\sin\theta$ and $\tan\theta$

θ is in quadrant I or IV

$x=1,\ y=\pm\sqrt{2},\ r=\sqrt{3}$

$\sin\theta = \pm\sqrt{\frac{2}{3}},\ \tan\theta = \pm\sqrt{2}$

4 Is $3\cos\theta + 4 = 2$ possible?

$3\cos\theta = -2$

$\cos\theta = -\frac{2}{3}$ which is possible.

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

Q. Right Angle Trigonometry - Assignment

1. Let be a point on the terminal side of θ Find the 6 trig functions of θ (Answers need not be rationalized).

a) $P(15,8)$

b. $P(-2,3)$

c. $P\left(-2\sqrt{5},-\sqrt{5}\right)$

2. If $\tan\theta=\frac{12}{5}$, θ in quadrant III, find $\sin\theta$ and $\cos\theta$

3. If $\csc\theta=\frac{6}{5}$, θ in quadrant II, find $\cos\theta$ and $\tan\theta$

4 $\cot\theta=\frac{-2\sqrt{10}}{3}$ find $\sin\theta$ and $\cos\theta$

5- Find the quadrants where the following is true: Explain your reasoning.

a. $\sin\theta>0$ and $\cos\theta<0$

b $\csc\theta<0$ and $\cot\theta>0$

c. all functions are negative

6. Which of the following is possible? Explain your reasoning.

a. $5\sin\theta=-2$

b. $3\sin\alpha+4\cos\beta=8$

c. $8\tan\theta+22=85$

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

R. Special Angles

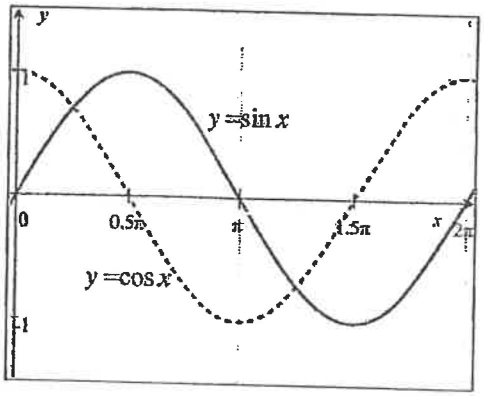
Students must be able to find trig functions of quadrant angles $(0,\ 90°, 180°, 270°)$ and special angles, those based on the $30°-60°-90°$ and $45°-45°-90°$ triangles.

First, for most calculus problems, angles are given and found in radians. Students must know how to convert degrees to radians and vice-versa. The relationship is 2π radians $= 360°$ or π radians $= 180°$ Angles are assumed to be in radians so when an angle of $\frac{\pi}{3}$ is given, it is in radians. However, a student should be able to picture this angle as $\frac{180°}{3} = 60°$ It may be easier to think of angles in degrees than radians, but realize that unless specified, angle measurement must be written in radians. For instance, $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

The trig functions of **quadrant angles** $\left(0,\ 90°, 180°, 270° \text{ or } 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right)$ can quickly be found. Choose a point along the angle and realize that r is the distance from the origin to that point and always positive. Then use the definitions of the trig functions.

θ	point	x	y	r	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0	$(1,0)$	1	0	1	0	1	0	does not exist	1	does not exist
$\frac{\pi}{2}$ or 90°	$(0,1)$	0	1	1	1	0	does not exist	1	does not exist	0
π or 180°	$(-1,0)$	-1	0	1	0	-1	0	does not exist	-1	does not exist
$\frac{3\pi}{2}$ or 270°	$(0,-1)$	0	-1	1	-1	0	Does not exist	-1	does not exist	0

If you picture the graphs of $y = \sin x$ and $y = \cos x$ as shown to the right, you need not memorize the table. You must know these graphs backwards and forwards.



- Without looking at the table, find the value of

a. $5\cos 180° - 4\sin 270°$

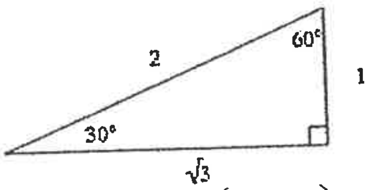
$$\boxed{\begin{array}{l} 5(-1)-4(-1) \\ -5+4=-1 \end{array}}$$

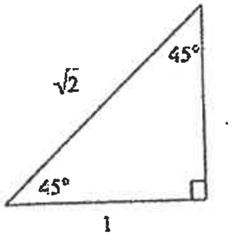
b $\left(\dfrac{8\sin\frac{\pi}{2} - 6\tan\pi}{5\sec\pi - \csc\frac{3\pi}{2}}\right)^2$ $\boxed{\left[\dfrac{8(1)-6(0)}{5(-1)-(-1)}\right]^2 = \left(\dfrac{8}{-4}\right)^2 = 4}$

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

Because over half of the AP exam does not use a calculator, you must be able to determine trig functions of **special angles**. You must know the relationship of sides in both $30° - 60° - 90° \left(\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}\right)$ and $45° - 45° - 90° \left(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}\right)$ triangles.





In a $30° - 60° - 90° \left(\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}\right)$ triangle, the ratio of sides is $1 - \sqrt{3} - 2$.

In a $45° - 45° - 90° \left(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}\right)$ triangle, the ratio of sides is $1 - 1 - \sqrt{2}$.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
$30°$ or $\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$45°$ or $\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60°$ or $\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Special angles are any multiple of $30° \left(\frac{\pi}{6}\right)$ or $45° \left(\frac{\pi}{2}\right)$ To find trig functions of any of these angles, draw them and find the **reference angle** (the angle created with the x-axis). Although most problems in calculus will use radians, you might think easier using degrees. This will create one of the triangles above and trig functions can be found, remembering to include the sign based on the quadrant of the angle. Finally, if an angle is outside the range of $0°$ to $360° (0$ to $2\pi)$, you can always add or subtract $360° (2\pi)$ to find trig functions of that angle. These angles are called **co-terminal angles**. It should be pointed out that $390° \neq 30°$ but $\sin 390° = \sin 30°$

- Find the exact value of the following

a. $4\sin 120° - 8\cos 570°$

Subtract 360° from 570°

$4\sin 120° - 8\cos 210°$

120° is in quadrant II with reference angle 60°

210° is in quadrant III with reference angle 30°

$$4\left(\frac{\sqrt{3}}{2}\right) - 8\left(\frac{-\sqrt{3}}{2}\right) = 6\sqrt{3}$$

b $\left(2\cos\pi - 5\tan\frac{7\pi}{4}\right)^2$

$(2\cos 180° - 5\tan 315°)^2$

180° is a quadrant angle

315° is in quadrant III with reference angle 45°

$$[2(-1) - 5(-1)]^2 = 9$$

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

S. Trigonometric Identities

Trig identities are equalities involving trig functions that are true for all values of the occurring angles. While you are not asked these identities specifically in calculus, knowing them can make some problems easier The following chart gives the major trig identities that you should know To prove trig identities, you usually start with the more involved expression and use algebraic rules and the fundamental trig identities. A good technique is to change all trig functions to sines and cosines.

Fundamental Trig Identities

$$\csc x=\frac{1}{\sin x},\quad \sec x=\frac{1}{\cos x},\quad \cot x=\frac{1}{\tan x},\quad \tan x=\frac{\sin x}{\cos x},\quad \cot x=\frac{\cos x}{\sin x}$$

$$\sin^2 x+\cos^2 x=1,\quad 1+\tan^2 x=\sec^2 x,\quad 1+\cot^2 x=\csc^2 x$$

Sum Identities

$$\sin(A+B)=\sin A\cos B+\cos A\sin B \qquad \cos(A+B)=\cos A\cos B-\sin A\sin B$$

Double Angle Identities

$$\sin(2x)=2\sin x\cos x \qquad \cos(2x)=\cos^2 x-\sin^2 x=1-2\sin^2 x=2\cos^2 x-1$$

- Verify the following identities.

1 $(\tan^2 x+1)(\cos^2-1)=-\tan^2 x$

$$(\sec^2 x)(-\sin^2 x)$$
$$\left(\frac{1}{\cos^2 x}\right)(-\sin^2 x)$$
$$-\tan^2 x$$

2. $\sec x-\cos x=\sin x\tan x$

$$\frac{1}{\cos x}-\cos x\left(\frac{\cos x}{\cos x}\right)$$
$$\frac{1-\cos^2 x}{\cos x}=\frac{\sin^2 x}{\cos x}$$
$$\sin x\left(\frac{\sin x}{\cos x}\right)=\sin x\tan x$$

3 $\dfrac{\cot^2 x}{1+\csc x}=\dfrac{1-\sin x}{\sin x}$

$$\left(\frac{\frac{\cos^2}{\sin^2 x}}{1+\frac{1}{\sin x}}\right)\frac{\sin^2 x}{\sin^2 x}=\frac{\cos^2 x}{\sin^2 x+\sin x}$$
$$\frac{1-\sin^2 x}{\sin x(1+\sin x)}=\frac{(1+\sin x)(1-\sin x)}{\sin x(1+\sin x)}$$
$$\frac{1-\sin x}{\sin x}$$

4. $\dfrac{1+\sin x}{\cos x}+\dfrac{\cos x}{1+\sin x}=2\sec x$

$$\left(\frac{1+\sin x}{\cos x}\right)\left(\frac{1+\sin x}{1+\sin x}\right)+\left(\frac{\cos x}{1+\sin x}\right)\left(\frac{\cos x}{\cos x}\right)$$
$$\frac{1+2\sin x+\sin^2+\cos^2 x}{\cos x(1+\sin x)}$$
$$\frac{1+2\sin x+1}{\cos x(1+\sin x)}=\frac{2+2\sin x}{\cos x(1+\sin x)}$$
$$\frac{2(1+\sin x)}{\cos x(1+\sin x)}=2\sec x$$

5 $\cos^4 2x-\sin^4 2x=\cos 4x$

$$(\cos^2 2x+\sin^2 2x)(\cos^2 2x-\sin^2 2x)$$
$$1[\cos 2(2x)]$$
$$\cos 4x$$

6. $\sin(3\pi-x)=\sin x$

$$\sin 3\pi\cos x-\cos 3\pi\sin x$$
$$0(\cos x)-(-1)\sin x=\sin x$$

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

R. Special Angles – Assignment

- Evaluate each of the following without looking at a chart.

1. $\sin^2 120° + \cos^2 120°$

2. $2\tan^2 300° + 3\sin^2 150° - \cos^2 180°$

3. $\cot^2 135° - \sin 210° + 5\cos^2 225$

4. $\cot(-30°) + \tan(600°) - \csc(-450°)$

5. $\left(\cos\frac{2\pi}{3} - \tan\frac{3\pi}{4}\right)^2$

6. $\left(\sin\frac{11\pi}{6} - \tan\frac{5\pi}{6}\right)\left(\sin\frac{11\pi}{6} + \tan\frac{5\pi}{6}\right)$

- Determine whether each of the following statements are true or false.

7 $\sin\frac{\pi}{6} + \sin\frac{\pi}{3} = \sin\left(\frac{\pi}{6} + \frac{\pi}{3}\right)$

8 $\dfrac{\cos\frac{5\pi}{3} + 1}{\tan^2\frac{5\pi}{3}} = \dfrac{\cos\frac{5\pi}{3}}{\sec\frac{5\pi}{3} - 1}$

9. $2\left(\frac{3\pi}{2} + \sin\frac{3\pi}{2}\right)\left(1 + \cos\frac{3\pi}{2}\right) > 0$

10. $\dfrac{\cos^3\frac{4\pi}{3} + \sin\frac{4\pi}{3}}{\cos^2\frac{4\pi}{3}} > 0$

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

S. Trig Identities – Assignment

- Verify the following identities.

1. $(1+\sin x)(1-\sin x)=\cos^2 x$

2. $\sec^2 x+3=\tan^2 x+4$

3. $\dfrac{1-\sec x}{1-\cos x}=-\sec x$

4. $\dfrac{1}{1+\tan x}+\dfrac{1}{1+\cot x}=1$

5. $\dfrac{\cos x-\cos y}{\sin x+\sin y}+\dfrac{\sin x-\sin y}{\cos x+\cos y}=0$

6. $\dfrac{\sin^3 x+\cos^3 x}{\sin x+\cos x}=1-\sin x\cos x$

7. $\csc 2x=\dfrac{\csc x}{2\cos x}$

8. $\dfrac{\cos 3x}{\cos x}=1-4\sin^2 x$

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

T. Solving Trig Equations and Inequalities

Trig equations are equations using trig functions. Typically they have many (or infinite) number of solutions so usually they are solved within a specific domain. Without calculators, answers are either quadrant angles or special angles, and again, they must be expressed in radians.

For trig inequalities, set both numerator and denominator equal to zero and solve. Make a sign chart with all these values included and examine the sign of the expression in the intervals. Basic knowledge of the sine and cosine curve is invaluable from section R is invaluable.

- Solve for x on $[0,2\pi)$

1 $x\cos x = 3\cos x$

Do not divide by $\cos x$ as you will lose solutions

$\cos x(x-3)=0$

$\cos x = 0$ $\quad x-3=0$

$x=\frac{\pi}{2},\frac{3\pi}{2}$ $\quad x=3$

You must work in radians.

Saying $x = 90°$ makes no sense.

2. $\tan x+\sin^2 x = 2-\cos^2 x$

$\tan x+\sin^2 x+\cos^2 x=2$

$\tan x+1=2$

$\tan x = 1$

$x=\frac{\pi}{4},\frac{5\pi}{4}$

Two answers as tangent is positive in quadrants I and III.

3 $3\tan^2 x-1=0$

$3\tan^2 x = 1$

$\tan^2 x=\frac{1}{3}$

$\tan x = \pm\sqrt{\frac{1}{3}}=\pm\frac{\sqrt{3}}{3}$

$x=\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}$

4. $3\cos x = 2\sin^2 x$

$3\cos x = 2\left(1-\cos^2 x\right)$

$2\cos^2 x+3\cos x-2=0$

$(2\cos x-1)(\cos x+2)=0$

$2\cos x = 1$ $\quad \cos x = -2$

$\cos x=\frac{1}{2}$ $\quad$ No solution

$x=\frac{\pi}{3},\frac{5\pi}{3}$

7 Solve for x on $[0,2\pi)$ $\frac{2\cos x+1}{\sin^2 x}>0$

$2\cos x=-1 \Rightarrow \cos x=\frac{-1}{2}\Rightarrow x=\frac{2\pi}{3},\frac{4\pi}{3}$

$\sin^2 x = 0 \Rightarrow x = 0,\pi$

+++++++0−−−−−−−∞−−−−−−−0+++++++

$0 \quad \frac{2\pi}{3} \quad \pi \quad \frac{4\pi}{3} \quad 2\pi$

Answer $\left[0,\frac{2\pi}{3}\right)\cup\left(\frac{4\pi}{3},2\pi\right)$

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

T. Solving Trig Equations and Inequalities - Assignment

- Solve for x on $[0,2\pi)$

1. $\sin^2 x = \sin x$

2. $3\tan^3 x = \tan x$

3. $\sin^2 x = 3\cos^2 x$

4 $\cos x + \sin x \tan x = 2$

5. $\sin x = \cos x$

6. $2\cos^2 x + \sin x - 1 = 0$

7. Solve for x on $[0,2\pi)$ $\dfrac{x-\pi}{\cos^2 x} < 0$

ODD PROBLEMS ONLY

Name: ____________________ Class: __________ Date: __________ Score: __________

U. Graphical Solutions to Equations and Inequalities – Assignment

- Solve these equations or inequalities graphically

1. $3x^3 - x - 5 = 0$

2. $x^3 - 5x^2 + 4x - 1 = 0$

3. $2x^2 - 1 = 2^x$

4. $2\ln(x+1) = 5\cos x$ on $[0, 2\pi)$

5 $x^4 - 9x^2 - 3x - 15 < 0$

6. $\dfrac{x^2 - 4x - 4}{x^2 + 1} > 0$ on $[0, 8]$

7. $x \sin x^2 > 0$ on $[0, 3]$

8. $\cos^{-1} x > x^2$ on $[-1, 1]$

ODD PROBLEMS ONLY

Tutorial Links

Topic A: **http://tinyurl.com/jmtcomp or http://tinyurl.com/hippofunct**
Topic B: **http://tinyurl.com/jmtdom or http://tinyurl.com/hippodom**
Topic C: **http://tinyurl.com/jmtgraphs or http://tinyurl.com/hippographs**

Topic D: **http://tinyurl.com/jmteven or http://tinyurl.com/jmteven2**
Topic E: **http://tinyurl.com/jmttransform or http://tinyurl.com/hippocompf**
Topic F: **http://tinyurl.com/jmtfactgroup or http://tinyurl.com/jmtfactsum**

Topic G: **http://tinyurl.com/jmtpointslope or http://tinyurl.com/tubeslopeint**
Topic H: ht**tp://tinyurl.com/jmtquad or http://tinyurl.com/tubequad**
Topic I: **http://tinyurl.com/jmtasym or http://tinyurl.com/hippoasymp**

Topic J: **http://tinyurl.com/jmtexp or http://tinyurl.com/tubexp**
Topic K: **http://tinyurl.com/jmtfrac or http://tinyurl.com/tubecomp**
Topic L: **http://tinyurl.com/jmtinverse or http://tinyurl.com/hippoinve**

Topic M: **http://tinyurl.com/jmtfracteqn or http://tinyurl.com/tubefreqn**
Topic N: **http://tinyurl.com/jmtabsol or http://tinyurl.com/hippoabso**
Topic O: **http://tinyurl.com/jmtquadin or http://tinyurl.com/tubequadin**

Topic P: **http://tinyurl.com/jmtsollog or http://tinyurl.com/hippolog**
Topic Q: **http://tinyurl.com/jmttrig or http://tinyurl.com/hippotrig**
Topic R: **http://tinyurl.com/jmtspec or http://tinyurl.com/hippospetrig**

Topic S: **http://tinyurl.com/jmtident or http://tinyurl.com/hippoident**
Topic T: **http://tinyurl.com/jmtsoltrig or http://tinyurl.com/hipposoltrig**
Topic U: **http://tinyurl.com/jmtsolsys or http://tinyurl.com/tubegraph**